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MECHANICS OF THE GYROSCOPE

THE DYNAMICS OF ROTATION

RICHARD F. DEIMEL

j lloyd dixon

MECHANICS OF THE GYROSCOPE

The Dynamics of Rotation

by Richard F. Deimel

Applications of gyroscopic phenomena are stressed in this elementary general treatment of the dynamics of rotation. Discussion of the subject begins with fundamentals; and while an understanding of college physics is assumed, no knowledge of vectors is required.

Velocity on a moving curve — rotating axes — acceleration of a point — Euler's equations — the inertia ellipsoid — and other fundamental topics are covered in the first three chapters. Important theorems are introduced, along with numerous practical exercises. The remainder of the book takes up gyroscopic phenomena and apparatus such as artificial horizons — free gyros — tops — motion of discs — the damped gyro — and many others. A 46-page chapter is devoted to the gyro-compass, while a final chapter discusses stabilizers as applied to ships and monorail vehicles.

"Remarkably concise and generous treatment of gyroscopic theory," INDUSTRIAL LABORATORIES. "This exceptional treatment of a complex subject well merits a place for this text in the laboratory of the practicing engineer," PROFESSIONAL ENGINEERING NEWS.

Index. 75 figures. 135 exercises. ix + 192pp. 5⅜ x 8.

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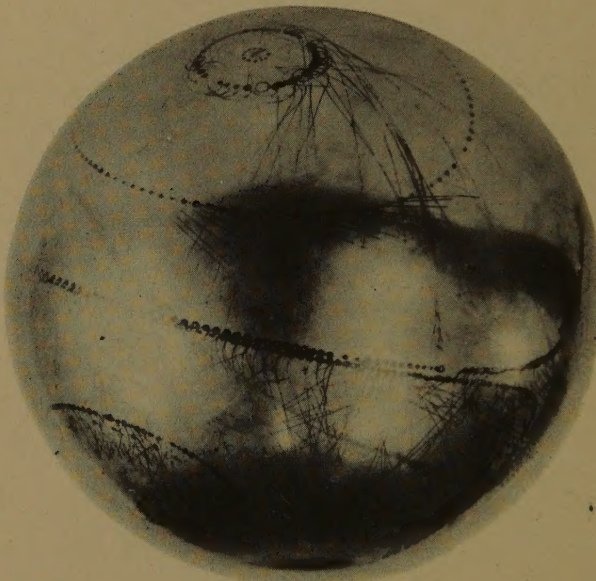
MECHANICS OF THE GYROSCOPE

MECHANICS



A steel ball, 4 $\pm$ 0.00005 inches in diameter, was supported and spun about a vertical axis by air jets in a hemispherical cup. At short intervals a pen was touched on the vertical axis until the ink dots formed a closed locus. The ball was displaced manually before each repetition of the experiment. These loci are polhodes; see Fig. 36. The photograph was made about 1929.

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GYROSCOPE

THE DYNAMICS OF ROTATION

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PREFACE

It has been known for many years that the mystifying properties of a spinning top or gyroscope could be applied to such a very practical end as finding the direction of true north. But it is only within the last two decades that they have been employed in operative devices for indicating the vertical and the horizontal, or any other desired direction, as well as the meridian, and that they have come into use for steering torpedoes and ships, and for stabilizing sea and aircraft and monorail vehicles. The development of high-speed machinery has brought to light unsuspected gyroscopic effects in rotating shafts, wheels, and propellers.

The action of a gyroscope ought to be no more astonishing than that of a gas. The two cases are very much alike in that the group behavior of each set of particles differs from that of the individual elements. The gas, indeed, is the more remarkable, because the motions of its molecules are perfectly disordered. It is only their uncountable number that gives the group its statistical properties. A dozen molecules per cubic foot would not be a gas. On the other hand, on account of their well-ordered motions, a few interconnected particles—for that matter even one—can be regarded as a perfectly respectable gyroscope.

The subject matter of this book is broader than its title indicates. It deals comprehensively with the dynamics of rotation. Since, however, translation offers no specially interesting problems or methods that do not occur also in rotation, it can be used as a text in general dynamics. Although the student is assumed to have gone through the mechanics given in college physics courses, the book starts at the very beginning of the subject. By departing from the historical order of development, by introducing exercises that deepen, not merely test, the student's knowledge, and by paying scant attention to such traditional elements as constant acceleration, simple harmonic motion, the pendulum, and computations of center of mass and moment of inertia, it was possible to take a broader point of view than is customary in textbooks for engineers, and to cover a large amount of ground in three short chapters. These topics, which properly

belong to physics or mathematics, have long been regarded as the simple things that prepare the reader for more advanced work. That the simple elements are fundamental, and are the bottom rungs of the ladder of learning, has the logical appeal that lies in the motto *Gradus ad Parnassum.* But it does not accord with my experience as student and teacher. I have found that it is more effective to go over these easy parts very quickly, with the least use of mathematics, and then to get a real insight into them as a by-product of working at the harder parts.

The remaining seven chapters deal with rotational phenomena and with the behavior of typical gyroscopic apparatus. The emphasis is on physical principles rather than on structural features that can be studied only at first hand, and it is hoped that the illustrative applications are given with sufficient detail so that the reader will be prepared to investigate the design and performance of any gyroscopic devices. Topics that are not likely to interest or to help the engineer have been omitted. Space might have been saved by using vector analysis, but this is not yet part of the engineer's mathematical equipment.

I have to thank Mr. Alexander Schein, Stabilizer Engineer, and Mr. Eric C. Sparling, Compass Engineer, of the Sperry Gyroscope Company, for generously putting their fund of practical information at my disposal, and my former student, Mr. Adrian Struyk, for his skilful preparation of the diagrams.

To the editors of the Engineering Science Series I am greatly indebted for helpful advice and patient editorial coöperation.

R. F. Deimel

Hoboken, N. J.
July, 1929

TO THE SECOND EDITION

Except for a new frontispiece and the correction of errata, the text has not been changed, as it still seems to serve the purpose for which it was prepared.

The title puts perhaps too much emphasis on the gyroscope. The book is an elementary general treatment of the dynamics of rotation, in which the most important applications arise necessarily from gyroscopic phenomena.

January 10, 1952

R. F. Deimel
Stevens Institute of Technology

CHAPTER V. THE FREE GYRO

CHAPTER VI. THE TOP

CHAPTER VII. MOTION ON A PLANE

CHAPTER VIII. CONSTRAINED MOTION

CHAPTER IX. THE GYRO-COMPASS

CHAPTER X. STABILIZERS

MECHANICS OF THE GYROSCOPE

MECHANICS OF THE GYROSCOPE

CHAPTER I

PLANE MOTION

1. Velocity on a Fixed Curve. The velocity v of any point P moving along a fixed curve, Fig. 1, is defined as

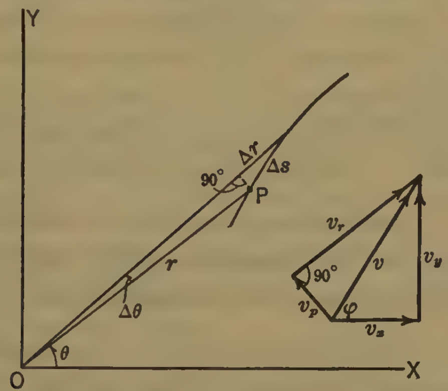


FIG. 1

$$v = \lim \frac{\Delta s}{\Delta t} = \frac{ds}{dt} = \dot{s};$$

v is tangent to the path of P.

For XY axes fixed to the curve the axial components of v are defined similarly:

$$v_x = \dot{x}, \qquad v_y = \dot{y}.$$

For polar coordinates

velocity along radius vector is $v_r = \lim \dfrac{\Delta r}{\Delta t} = \dot{r}$,

velocity perp. to radius vector is $v_p = \lim \dfrac{r\Delta\theta}{\Delta t} = r\dot{\theta} = r\omega$,

where ω, measured in radians per second, is the angular velocity of the radius vector.

The velocity v is a vector tangent to the path of P. It obeys the law of vector addition because linear displacements, from which velocities are derived, are added or superposed vectorially, *i.e.*, according to the parallelogram law.

Ex. 1. Show that $\frac{d}{dt}(r \cos \theta) = v \cos \varphi$.

2. Velocity on a Moving Curve. If the curve in Fig. 1 is in motion, P will have two simultaneous displacements: its displacement along the curve and its displacement with the curve. That along the curve is called relative, and that with the curve is the displacement of constraint. The displacement of constraint is due to the motion of that point of the curve at which P happens to be at any instant. This leads to the following result.

THEOREM I. ***The velocity of any point P on a moving curve is the vector sum of the relative velocity $\dot{s}$ and the velocity of constraint (velocity of the coincident point of the curve).***

The velocity of constraint is found thus: if the curve and the axes, and therefore O, translate (without turning) at the rate u, the velocity of the coincident point of the curve at P is u. If the curve rotates about O at the rate ω, the velocity of the coincident point is $r\omega$, and if this rotation occurs simultaneously with the translation u, the coincident point will have both motions. This gives the following theorem.

THEOREM II. ***The velocity of constraint of any point of a rigid body is the vector sum of the velocity of any other point O, and of $r\omega$ due to the rotation of the coincident point at P about O; $r\omega$ is perpendicular to OP.***

Ex. 2. In the connecting-rod mechanism, Fig. 2, the velocity u of the crank-pin O is given; find that of the cross-head C.

If the moving axes are taken as fixed to OC, the relative velocity of C is zero. The velocity of constraint of C is v, the resultant of u and a velocity $l\omega$ perpendicular to OC, by Theorem II. But v is along the line of stroke (horizontal), hence the construction.

Ex. 3. A particle of water moves with relative velocity v along the vane of a turbine rotor turning at n r.p.m. Find its velocity.

The velocity of constraint at P is $R\omega$ where $\omega = 2\pi n/60$ rad. per sec. The velocity of the particle P is the resultant of v (tangent to vane) and $R\omega$. If O is taken as the moving origin, Theorem II states that $R\omega$ is the resultant of u and $r\omega$.

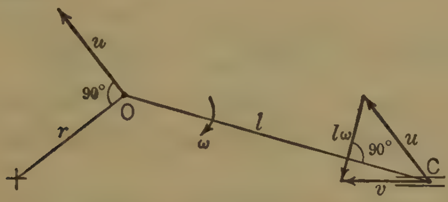


FIG. 2

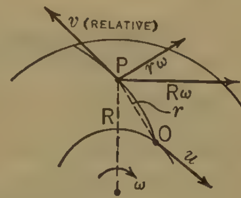


FIG. 3

Ex. 4. A 6-ft. wheel rolls so that its center goes at 5 ft./sec. while a point 2 ft. from the center travels along a spoke at 30° to the horizontal at 1 ft./sec. Find the velocity of the point.

If ω is the angular velocity of the wheel the linear velocity of the lowest point (point of contact) is $5 - 3\omega$. In order that this point may not slip along the ground, $5 - 3\omega = 0$.

Ex. 5. In Fig. 4 a taut string is being unwound from a stationary spool. Find the velocity of P.

In an infinitesimal displacement P moves to Q in time dt; in the limit the velocity of P is $l\dot{\theta}$ normal to l.

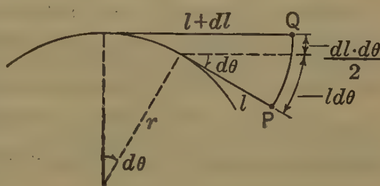


FIG. 4

Ex. 6. Cartesian axes are fixed to a moving curve. Show that if u_0, v_0 are the velocity components of the origin and ω the angular velocity of the axes, the velocities of any particle (x, y) moving along the curve are

$$u = u_0 + \dot{x} - y\omega, \quad v = v_0 + \dot{y} + x\omega.$$

3. Acceleration. When a velocity changes in magnitude and direction from v at P to $v + dv$ at Q in time dt, Fig. 5, the infinitesimal vector change of v is δv, not dv. By definition the acceleration a at P is the vector time-rate of change of velocity:

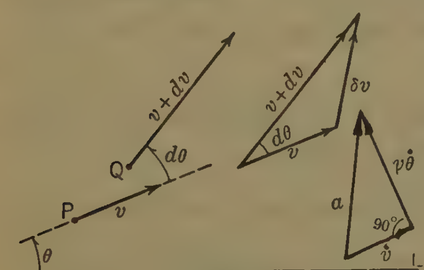


FIG. 5

$$a = \frac{\delta v}{dt}.$$

Accelerations obey the law of vector addition.

From the velocity triangle in Fig. 5, we have

$$\textit{acceleration along } v \textit{ is } \frac{(v + dv)\cos d\theta - v}{dt} = \frac{dv}{dt} = \dot{v},$$

$$\textit{acceleration perp. to } v \textit{ is } \frac{(v + dv)\sin d\theta}{dt} = v\frac{d\theta}{dt} = v\dot{\theta}.$$

Here $\dot{v}$ is due to the change of magnitude of v, and $v\dot{\theta}$ arises from the change of direction of v; their vector sum is the resultant acceleration a as shown.

This result is fundamental, and the following demonstration which is, except in notation, the method of vector analysis, will not be superfluous.

Velocity is a vector; its derivative dv/dt is not its vector rate of change, because the increments studied in the infinitesimal calculus are scalar quantities having only magnitude but not direction. But dv/dt is the acceleration when the direction of v is constant. Thus in Fig. 5 the projection $v \cos \theta$ of v on any line L is constant in direction and its derivative is the acceleration a_L along L:

$$a_L = \frac{d}{dt}(v \cos \theta) = \dot{v} \cos \theta - v\dot{\theta} \sin \theta;$$

compare Ex. 1. When $\theta = 0$ and $\theta = -90°$, L lies respectively along v and perpendicular to v in the sense of $\dot{\theta}$ (toward the left in Fig. 5), and we get the above results.

Theorem III. ***The acceleration due to scalar change of v is $\dot{v}$ along v; that due to the rotation of v at the rate ω is $v\omega$ normal to v in the sense of ω.***

If ρ is the radius of curvature of the path at P in Fig. 5 and $PQ = ds$, then $\rho d\theta = ds$ or $\rho\dot{\theta} = v$. Hence

$$\textit{acceleration tangent to path is } \dot{v} = \ddot{s},$$

$$\textit{acceleration toward center of curvature is } \frac{v^2}{\rho} = \rho\dot{\theta}^2.$$

Consider the polar axes in Fig. 1. The accelerations due to scalar changes of v_r and v_p are $\dot{v}_r = \ddot{r}$ and $\dot{v}_p = r\dot{\omega} + \dot{r}\omega$; those due to rotation are $+ v_r\omega$ and $- v_p\omega$, the latter being negative because it is directed toward O, *i.e.*, in the sense of decreasing r. Hence

$$\textit{acceleration along radius vector is } a_r = \dot{v}_r - v_p\omega = \ddot{r} - r\omega^2,$$

$$\textit{acceleration perp. to radius vector is } a_p = \dot{v}_p + v_r\omega = \frac{1}{r}\frac{d}{dt}(r^2\omega).$$

The right-hand members of these equations are correct only when the curve is fixed, for by § 2, v_r is not $\dot{r}$ and v_p is not $r\omega$ if the curve itself is in motion. But the left-hand members are correct provided v_r and v_p represent the actual velocities of P and provided both turn at the rate ω; see Ex. 17.

Ex. 7. The polar equation of a conic of eccentricity ϵ and semi-parameter l is $r = l(1 + \epsilon \cos \theta)$ with the origin at the right-hand focus. Find $\dot{r}$, $\ddot{r}$ when $\theta = 0$. Note that $\dot{r} = 0$ does not make $\ddot{r}$ vanish.

Ex. 8. Find the acceleration of P in Fig. 4; see Ex. 5.

$l\omega$ turns at the rate ω and produces $l\omega^2$ along l toward the point of tangency. The scalar rate of $l\omega$ is $l\dot{\omega} + \dot{l}\omega$ along $l\omega$. The term $\dot{l}\omega$ is not to be regarded as due to the rotation of a velocity $\dot{l}$, because P has no velocity along l. It arises from the lengthening of the string.

Ex. 9. Solve Ex. 8 by projecting the vector difference between $l\omega$ at P and $(l + dl)$ $(\omega + d\omega)$ at Q, along and normal to l. Note that the projection of the vector difference is the difference of the projections of the vectors.

Ex. 10. Find the acceleration of any point on the circumference of a rolling circle whose center has a velocity u.

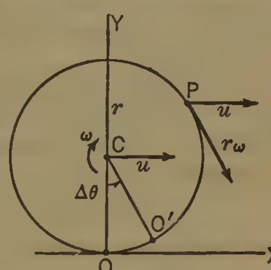


FIG. 6

By Theorem II, p. 2, P has the two velocities shown. Only $r\omega$ turns. If u, ω are constant, the acceleration of P is $(r\omega)\omega$. This is so for every point on the circumference and therefore for O although, as in Ex. 4, the velocity of O is zero in rolling without slipping.

Now if O has acceleration, it must have displacement, since the velocity of a point cannot change unless the point changes its position. To find it, use the equations of the cycloidal path of P:

$$x = r(\theta - \sin \theta), \quad y = r(1 - \cos \theta), \quad \theta = O'CP.$$

When P is at O', $\theta = \Delta\theta$ and

$$x = r(\Delta\theta - \sin\Delta\theta) = 0 \quad \textit{to terms of second order},$$
$$y = r(1 - \cos\Delta\theta) = r(1 - 1 + \tfrac{1}{2}\Delta\theta^2 + \cdots)$$
$$= \tfrac{1}{2}r\Delta\theta^2 \quad \textit{to terms of second order.}$$

The vertical displacement is thus the same as if O had moved for a time Δt with constant acceleration $r(\Delta\theta/\Delta t)^2$ according to the familiar law $s = \frac{1}{2}at^2$. This may be proved in another way. By Taylor's expansion, any displacement s may be written in the form

$$s = s_0 + \dot{s}_0 t + \frac{\ddot{s}_0 t^2}{\underline{|2}} + \cdots,$$

where s_0, $\dot{s}_0$, $\cdots$ are the values of s, $\dot{s}$, $\cdots$ when $t = 0$. Hence when a point starts at $s = 0$ with initial velocity $\dot{s}_0 = 0$ and moves for a *small* time t, $s = \frac{1}{2}\ddot{s}_0 t^2$ for the beginning of any small motion. For the point O', $\ddot{s}_0 = r(\Delta\theta/\Delta t)^2$ and $t = \Delta t$.

Ex. 11. The acceleration perpendicular to the radius vector is

$$\frac{1}{r}\frac{d}{dt}(r^2\omega) = 2\dot{r}\omega + r\dot{\omega}.$$

Explain the '2.'

The rotation of $\dot{r}$ produces $\dot{r}\omega$. At times t and $t + dt$ the velocities *perpendicular* to r are $r\omega$ and $(r + dr)(\omega + d\omega)$; the acceleration due to this change of velocity is $\dot{r}\omega + r\dot{\omega}$, which, added to the $\dot{r}\omega$ arising from the rotation of $\dot{r}$, gives the result stated above.

Ex. 12. Can a point describing a circle at constant speed have radial acceleration directed *away* from the center?

No matter whether the point goes right or left around the center, the radial acceleration is $r\omega^2$ toward the center. An increase in ω increases $r\omega^2$, *i.e.*, adds an inward acceleration; a decrease in ω decreases $r\omega^2$ and in effect adds an outward acceleration.

If ω is decreased by ϵ, the rotations $+\,\omega$ and $-\,\epsilon$ produce velocities $+\,r\omega$ and $-\,r\epsilon$. But $+\,r\omega$ turns at the rates $+\,\omega$, $-\,\epsilon$ since the point has both rotations; likewise $-\,r\epsilon$ turns at $+\,\omega$, $-\,\epsilon$. The resultant acceleration toward the center is

$$r\omega^2 + r\epsilon^2 - 2r\omega\epsilon = r(\omega - \epsilon)^2,$$

where $2r\omega\epsilon$ points away from the center.

4. Acceleration on a Moving Curve. In Fig. 7 the axes are attached to the curve; they rotate at the rate ω and their origin translates at velocities u_0, v_0. The curve is always assumed to be rigid.

The relative velocities of P are $\dot{x}$, $\dot{y}$, and the velocities of

constraint, Theorem II, p. 2, are $u_0 - y\omega$, $v_0 + x\omega$; hence by Theorem I the resultant velocities are

$$u = u_0 + \dot{x} - y\omega, \qquad v = v_0 + \dot{y} + x\omega.$$

Since u and v are always parallel to the axes and therefore *turn* with them, Theorem III, p. 4, gives for the accelerations of P

$$\dot{u} - v\omega, \qquad \dot{v} + u\omega.$$

Hence

$$a_x = (\dot{u}_0 - v_0\omega - y\dot{\omega} - x\omega^2) + \ddot{x} - 2\dot{y}\omega,$$
$$a_y = (\dot{v}_0 + u_0\omega + x\dot{\omega} - y\omega^2) + \ddot{y} + 2\dot{x}\omega.$$

When x, y are constant, the parentheses are the accelerations of P regarded as *fixed* to the axes; they are therefore the accelerations of *constraint.* The terms $\dot{u}_0 - v_0\omega$, $\dot{v}_0 + u_0\omega$ are the accelerations of the *origin,* and $-y\dot{\omega} - x\omega^2$, $x\dot{\omega} - y\omega^2$ are those of P produced only by *rotation.* Here $\dot{\omega}$ is the *angular* acceleration of the whole system; $\ddot{x}$, $\ddot{y}$ are due to the *relative* motion.

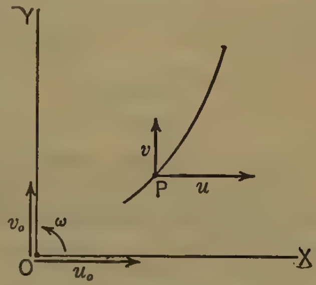


FIG. 7

The components $-2\dot{y}\omega$, $+2\dot{x}\omega$ are called the *Coriolis* accelerations. The Coriolis acceleration is twice the product of the relative velocity by the angular velocity of the rotating path; it is normal to the relative velocity in the sense of ω, and exists whenever the velocity along a curve is turned by the rotation (not the curvature) of the curve. The coefficient 2 is explained as in Exs. 11, 12.

The interpretation of a_x, a_y may be stated as Theorems IV, V.

THEOREM IV. *The acceleration of constraint of any point P on a moving rigid system is the resultant of*

(i) *the acceleration of any other point O,*

(ii) $r\omega^2$ *from P to O, due to the rotation of* $OP = r$ *about O,*

(iii) $r\dot{\omega}$ *perpendicular to OP, due to the acceleration* $\dot{\omega}$ *of OP.*

THEOREM V. *The acceleration of a particle on a moving rigid curve is the resultant of*

(i) *the acceleration of constraint, Theorem IV,*

(ii) *the relative acceleration: found as if the curve were at rest,*

(iii) *the Coriolis acceleration: twice the product of the relative velocity of the particle along the curve by the angular velocity of the curve; it points in the sense in which ω turns the arrow-head of the relative velocity vector.*

Ex. 13. Find the acceleration of C in Ex. 2.

Since the acceleration of C is along the line of stroke, Theorem IV gives the construction in Fig. 8: u^2/r is along r, $l\omega^2$ and $l\dot{\omega}$ respectively along and perpendicular to l; u is assumed constant.

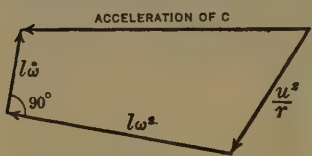


FIG. 8

Ex. 14. Show that the acceleration of the lowest point of the wheel in Ex. 4 is 8.33 ft./sec.² vertically upward; see Ex. 10.

Ex. 15. If the origin in Fig. 1 has accelerations ρ and p along and perpendicular to r, show that the accelerations of P are $\rho + \ddot{r} - r\omega^2$, $p + (1/r)d(r^2\omega)/dt$.

Ex. 16. Find the acceleration of P in Ex. 3.

The accelerations are: constraint, $R\omega^2$ toward center, $R\dot{\omega}$ normal to R; relative, $\dot{v}$ along v, v^2/ρ along radius of curvature ρ; Coriolis, $2v\omega$ normal to v toward the right.

Ex. 17. The radius vector r to a point P moving on a curve makes an angle θ with a radius vector fixed to the curve, the attached radius being inclined at φ to the horizontal. Derive the accelerations of P from Theorems IV, V and show that they reduce to

$$a_r = \ddot{r} - r(\dot{\theta} + \dot{\varphi})^2, \quad a_p = \frac{1}{r}\frac{d}{dt}[r^2(\dot{\theta} + \dot{\varphi})].$$

The curve turns about a fixed origin at the rate $\dot{\varphi}$.

5. Instantaneous Center. The instantaneous center (abbreviated to I.C.) of a rigid body having plane motion is defined as that point whose velocity is zero at a given instant. The instantaneous axis goes through the I.C. and is perpendicular to the plane of motion. The slight distinction between instantaneous center and axis does not exist in the case of space motion.

Since the body is rigid, the velocity of any point is the velocity of constraint, Theorem II, p. 2, and is normal to the radius

vector from the I.C. Thus, the I.C. of the connecting-rod in Fig. 2 is at the intersection of the crank and the normal to the line of stroke at the cross-head.

The I.C. does not remain fixed, but continually changes its position in space and in the body itself. In Fig. 9, when any point P of the body moves to P', suppose that the I.C. changes from C to C'. The velocities of P and P' are normal to PC and $P'C'$. Now $v_r = 0$ at P, but when P moves to P', v_r changes by an amount $dv_r = d\dot{r}$. Let a_r, a_p be the accelerations of P along and perpendicular to r, and put $\dot{\theta} = \omega$. The projections on a_r, a_p of the vector velocity differences are

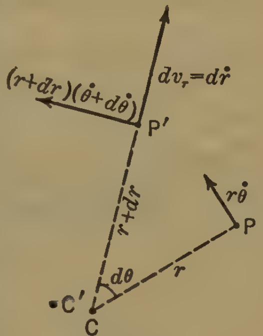


FIG. 9

$$a_r dt = d\dot{r} \cos d\theta - (r + dr)(\omega + d\omega) \sin d\theta,$$
$$a_p dt = (r + dr)(\omega + d\omega) \cos d\theta + d\dot{r} \sin d\theta - r\omega.$$

Putting $\sin d\theta = d\theta$ and $\cos d\theta = 1$, and rejecting second order infinitesimals, we get, with the instantaneous center at the origin,

$$a_r = \ddot{r} - r\omega^2, \qquad a_p = \frac{d}{dt}(r\omega).$$

If the I.C. is not at the origin, $a_p = 2\dot{r}\omega + r\dot{\omega}$, p. 5; if it is, $a_p = \dot{r}\omega + r\dot{\omega}$. Half of the Coriolis acceleration is missing in the second case because the relative velocity $\dot{r}$ of P is zero; half is present because r changes by dr in time dt; see Ex. 11.

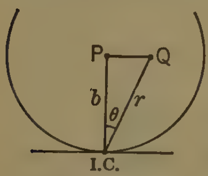


Fig. 10

Ex. 18. Find the acceleration of the center and of the I.C. of a circle rolling on a straight line.

In Fig. 10 the point of tangency is the I.C.; see Ex. 4. Since b is constant it cannot be used as the radius vector. Deal therefore with some point Q on the horizontal path of P. From $r = b \sec \theta$ find $\dot{r}$, $\ddot{r}$, put $\dot{\theta} = \omega$, $\theta = 0$, substitute in the equations and get $a_r = 0$, $a_p = b\dot{\omega}$ as the accelerations of P.

By Theorem IV, p. 7, the acceleration of the lowest point of the circle has components $b\dot{\omega}$, $-b\dot{\omega}$, $b\omega^2$; see Ex. 10.

6. Centrodes. The locus of the I.C. in space is called the space centrode; the locus of the I.C. in the body, the body centrode. For example, the line on which a circle rolls is the space centrode, the circle itself being the body centrode; the cycloidal path of a point on the circumference is not a centrode.

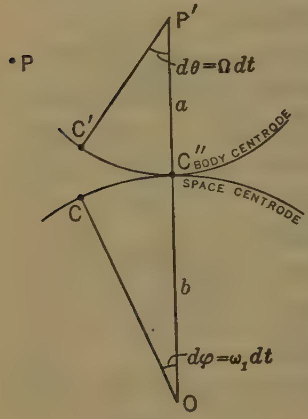


Fig. 11

In Fig. 11, C'' is the I.C. at time t; dt seconds earlier, when a point P' was at P, the I.C. was at C, and C' on the body was then also at C.

Since C'' on the body is the I.C. its velocity is zero; therefore C'' on the body does not *slip* along the space centrode. That is, the body centrode *rolls* on the space centrode.

For rolling without slip, $CC'' = C'C''$; hence if O, P' are the centers of curvature of the centrodes,

$$a d\theta = b\, d\varphi, \quad \text{or} \quad a\Omega = b\omega_1.$$

Ω is called the *relative* angular velocity of rolling because it is the velocity of the body relative to the moving line of centers $P'O$. The angle actually turned through by the body in time dt is

$$PC'P' = d\varphi + d\theta,$$

since in the limit the points O, C, C', P lie on a straight line; that is, CC', as will be seen below, is an infinitesimal of order higher than the first.

The absolute velocity ω of rolling is thus

$$\omega = \omega_1 + \Omega,$$

where ω is the rate at which the body is turning about C''.

It is not often necessary to know the acceleration a_i of the I.C., but several ways of finding it will be given on account of the importance of the kinematical reasoning.

(i) When C is the I.C. its velocity is zero; dt later, when C has moved to C', its velocity has increased to

$$(C''C)\omega = bd\varphi\omega \quad \text{or} \quad a_i = b\omega\omega_1 = a\omega\Omega.$$

(ii) In the motion from C to C' the initial velocity is zero; hence as in Ex. 10, p. 5,

$$CC' = \tfrac{1}{2}a_i dt^2$$

which, as was mentioned above, is of the second order. Since the infinitesimal arcs CC'', $C'C''$ are ultimately circular, we have

$$\text{angle } CC''C' = \tfrac{1}{2}(d\varphi + d\theta);$$

hence

$$CC' = C''C'\frac{d\varphi + d\theta}{2}, \quad \text{or} \quad a_i = b\omega\omega_1 = a\omega\Omega.$$

(iii) The acceleration of C'' may be found from that of P' by Theorem IV, p. 7: *acceleration of C'' perpendicular* to $P'C''$ *is*

$$(a + b)\dot{\omega}_1 - a\dot{\omega},$$

which vanishes since $(a + b)d\varphi = a(d\varphi + d\theta)$.

The acceleration along $P'C''$ is the acceleration of P' minus the acceleration of C'' due to rotation of $P'C''$ at rate ω.

Hence

$$a_i = (a + b)\omega_1^2 - a\omega^2 = -b\omega\omega_1 = -a\omega\Omega,$$

the negative sign showing the direction to be from C'' to P'.

Ex. 19. Draw Fig. 11 when the body centrode rolls on the concave side of the space centrode. Take $a < b$ and show that $\omega = \Omega - \omega_1$ when ω and Ω are opposite to ω_1.

Ex. 20. In Ex. 19 take $a > b$ and show that $\omega = \omega_1 - \Omega$ when ω and ω_1 are opposite to Ω.

Ex. 21. A man walks through a train with velocity v while the train runs at u and rounds a curve of radius R. Use Theorem V, p. 8, to find his acceleration. Neglect the rotation of the earth.

CHAPTER II

MOTION IN SPACE

7. Angular Velocity of a Rigid Body. In Fig. 12, imagine the XY plane attached to a rigid body that rotates at ω radians per second about a line I fixed in space, XIY being coplanar. The velocity v of any point P on X is $p\omega$. Hence

$$v = x\omega \sin \varphi.$$

Thus v is the same as if the X axis, *and therefore the body,* rotated about Y at the rate $\omega \sin \varphi$. For this reason $\omega \sin \varphi$ is called the angular velocity of the body about Y. The angular velocity of the body about X is shown in the same way to be $\omega \cos \varphi$. Hence if ω, about I, is regarded as a vector laid off along I, its axial projections will represent the simultaneous angular velocities of the body about the axes.

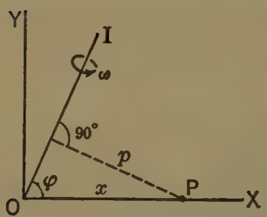


FIG. 12

Conversely, if the body has simultaneous velocities ω_1 about X and ω_2 about Y, any point (x, y) will have linear velocities

$$y\omega_1 \text{ \textit{out of the page,}} \qquad x\omega_2 \text{ \textit{into the page.}}$$

If the point is to be momentarily at rest

$$x\omega_2 = y\omega_1,$$

which defines a straight line through the origin; this is the *instantaneous axis or the resultant angular velocity axis.*

Ex. 22. Prove by the method used in the text that the angular velocity about the instantaneous axis is $\sqrt{\omega_1^2 + \omega_2^2}$.

Ex. 23. It is shown in the text that angular velocity obeys the parallelogram law for orthogonal vectors. Show that if the law holds for a rectangle it is true for any parallelogram.

Combine part of one vector with all of the other.

The foregoing considerations give the following theorem.

THEOREM VI. ***Angular velocity is a vector obeying the law of vector addition.***

The following convention of signs (sense) will be used:

Point the vector in the direction in which a right-hand screw would be advanced by the angular velocity.

Theorem VI is evidently not restricted to coplanar vectors. If XYZ axes are turning about any axis I through the origin the components of ω are

$$\omega_1 = \omega \cos (X, I), \quad \omega_2 = \omega \cos (Y, I), \quad \omega_3 = \omega \cos (Z, I).$$

Ex. 24. A man stands in latitude θ. At what rate is he turning about his zenith axis? about a tangent to his meridian?

Ex. 25. The letter Y, the angle at the fork being 60°, is spun about its stem. At what rate is the left branch turning about the right?

The entire Y, and therefore the left branch, is spinning about the right branch at $\omega \cos 30°$, the component $\omega \sin 30°$ being *perpendicular* to the right branch. If ω is resolved into components along both branches each is $(\omega/2) \sec 30°$ *but this is not the whole spin about a branch.*

Ex. 26. Show that the linear velocity of any point (x, y) on the body in Fig. 12 is $y\omega_1 - x\omega_2$ normal to the paper.

When a body turns about a fixed point O, the axis of rotation is the straight line through O and all other points of zero velocity. Any translation may be superposed on the rotation.

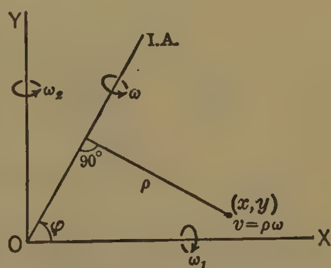


FIG. 13

In Fig. 13, XY are fixed in space (not in the body as in Fig. 12) and coplanar with $I.A.$; O is either fixed or momentarily at rest; ρ is any normal to $I.A.$ in the XY plane.

The linear velocity of a point (x, y) at the end of ρ is $v = \rho\omega$. The angular velocity of the *point* (x, y) is

$$v/\rho \text{ about } I.A., \qquad -v/y \text{ about } X, \qquad +v/x \text{ about } Y.$$

Only v/ρ is also the angular velocity of the *body;* the others are not, because they vary with the length of ρ and do not agree with the definitions in Art. 7.

Ex. 27. Prove that all points lying on a straight line through the origin have the same angular velocity about X.

Ex. 28. Prove that all points on the instantaneous axis have the same angular velocity about X or Y as the body has.

This result is important. It is really a definition of the angular velocity components of a body.

8. Moving Axle. An axle, like the axle of a wheel, is an axis upon which a body is mounted with freedom of rotation. Unless there is friction, rotation of the axle about its own geometric axis cannot turn the body. *Rotation of the axle about an oblique*

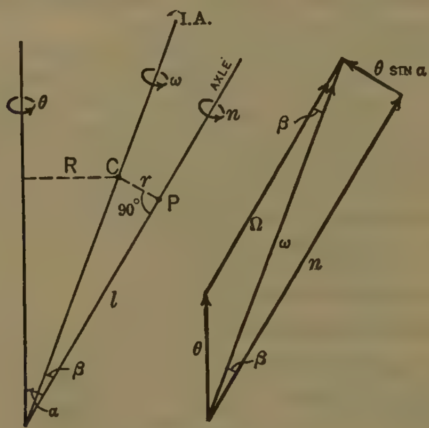


Fig. 14

axis is the cause of gyroscopic phenomena. To study the motion, assume the axle to be turning about a fixed point on it with angular velocity θ (not $\dot{\theta}$), as in Fig. 14.

Before the axle is allowed to move, let the rotor (the body) be given a spin of n radians per second; n is the absolute axle spin and cannot be changed by motion of the axle unless there is friction. This means that $\theta \cos \alpha$ does not contribute to the velocity of the rotor about the axle.

Let Ω about the axle be the angular velocity of the body relative to the plane through the θ and n axes. Since $\theta \cos \alpha$ is the axle component of the velocity of this plane, and since Ω is relative to it, the absolute axle velocity of the rotor is

$$n = \theta \cos \alpha + \Omega.$$

The resultant of θ and Ω is the absolute velocity ω of the rotating body. Since ω is the whole or resultant velocity of the rotor, the ω axis has no velocity and is therefore the instantaneous axis. It lies in the (θ, n) plane; its locus, a cone about the θ axis, is the *space centrode*, and its locus in the body, a cone about the axle, is the *body centrode*. As on p. 10, the latter cone rolls on the former; Ω is called the *relative*, and ω the *absolute* velocity of rolling.

The absolute velocity ω is shown in the velocity triangle θ, Ω, ω. It can be found in another way. The resultant spin about the axle is n; $\theta \sin \alpha$ (normal to axle) turns the rotor with the axle but $\theta \cos \alpha$ cannot do so without the aid of friction which is excluded at present. Therefore ω is the resultant of n and $\theta \sin \alpha$ as shown.

The law of sines applied to the velocity triangles in Fig. 14 gives

$$\theta : n : \Omega : \omega = \tan \beta : \sin \alpha : \frac{\sin (\alpha - \beta)}{\cos \beta} : \frac{\sin \alpha}{\cos \beta}.$$

Ex. 29. Obtain the above relations by considering the linear velocities of P in Fig. 14.

The velocity of P can be expressed in these equivalent forms:

$\theta(l \sin \alpha)$ *due* to θ; $\omega(r \cos \beta)$ *due* to ω; nr *due* to n

because the circle of radius PC, center P, is rolling with velocity n along a line through C tangent to the circle.

Equating these values gives some of the required ratios. Another may be found as follows.

The velocity of C as a point on the rotor (not on the $I.A.$) is θR due to θ alone. But C, on the rotor, has also the relative velocity Ωr due to Ω alone. Their resultant, due to θ and Ω simultaneously, must be zero in order that C may lie on the $I.A.$; hence $\theta R = \Omega r$.

Ex. 30. A bevel gear of semi-angle 45° is held fixed while another of semi-angle 30° rolls around it at m r.p.m. Find the absolute angular velocity of the latter.

The axle makes m r.p.m.; hence $\theta = m$; $\alpha = 75°$, $\beta = 30°$. Notice that θ/Ω is the velocity ratio of the gears when their axles remain fixed.

Ex. 31. Draw the velocity triangles in Fig. 14 when

(a) the axle lies between ω, θ,

(b) θ lies between Ω, ω; see Exs. 19, 20.

Ex. 32. A hoop of radius r is inclined at φ to the vertical and rolls around a horizontal circle, the center of the hoop describing a circle of radius R at velocity v. Find the angular velocities of the hoop.

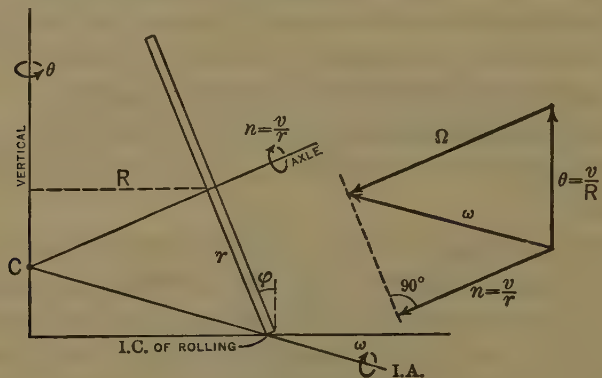


Fig. 15

The hoop turns about a vertical as shown in Fig. 15. The $I.A.$ passes through the intersection C of the axle and the vertical, and through the instantaneous center of rolling, because these points have zero velocity. As in Ex. 29, $nr = v$; ω is the hypotenuse of a right triangle one side of which is n; $\theta R = v$: see Ex. 28.

9. Angular Acceleration. The vector time-rate of change of angular velocity is called angular acceleration. We shall find it by the method used in deriving Theorem III, p. 4.

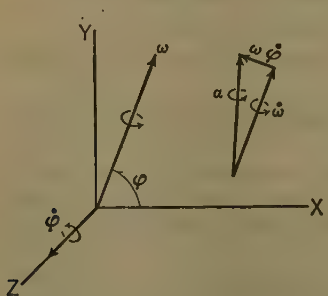


Fig. 16

Figure 16 shows any angular velocity vector ω moving in the XY plane at the rate $\dot{\varphi}$ about Z; XYZ are fixed in space. By analogy with a_L, p. 4, α_x is called

the angular acceleration about X, where

$$\alpha_x = \frac{d\omega_x}{dt} = \frac{d}{dt}(\omega \cos \varphi)$$
$$= \dot{\omega} \cos \varphi - \omega\dot{\varphi} \sin \varphi.$$

As on p. 4, α_x is the projection on X of the angular acceleration vector α whose components are, Fig. 16,

$\dot{\omega}$ along and positive in the sense of ω,

$\omega\dot{\varphi}$ perpendicular to ω in the sense of $\dot{\varphi}$, *i.e.*, in the direction toward which $\dot{\varphi}$ turns the head of the ω vector.

This gives the following result.

THEOREM VII. ***The angular acceleration due to scalar change of angular velocity is $\dot{\omega}$ along ω; that due to the rotation of the ω vector at the rate $\dot{\varphi}$ is $\omega\dot{\varphi}$ orthogonal to ω and $\dot{\varphi}$, in the direction toward which $\dot{\varphi}$ turns the head of the ω vector.***

In order to have a consistent way of drawing angular velocity and angular acceleration vectors we shall use the following plan.

CONVENTION OF SIGNS. ***Point the angular vectors so that they will represent clockwise rotation when viewed from tail toward head.***

This system is called *right-handed* because the rotation will advance a right-handed screw along the vector. The same convention was used on page 13. Note that there is no sense of rotation associated with $\omega\dot{\varphi}$; but it has *direction*, which is determined as stated in Theorem VII. The sense of rotation of $\dot{\omega}$ is the same as that of ω provided $\dot{\omega}$ is positive or corresponds to an increase of ω.

Ex. 33. When viewed from above, a wheel is turning counterclockwise about a vertical axle; it starts at n r.p.m. and slows down at the rate of m rev. per sec. per min. while the top of the axle turns eastward at p r.p.m. Find its angular acceleration.

All angular quantities should be expressed in terms of radians. By Theorem VII and the convention of signs above, the acceleration produced by the *decrease* of angular velocity is represented by a *downward* vector of

magnitude $\frac{2\pi m}{60}$ rad. per sec.²; that produced by the rotation of the axle is $\frac{2\pi n}{60} \cdot \frac{2\pi p}{60}$ rad. per sec.² and is directed toward the east because the arrowhead of the n vector, which points upward, is turned eastward by p.

10. Rotating Axes. In Fig. 12 the angular velocity vector $\omega \sin \varphi$, along Y, turns about X at the rate $\omega \cos \varphi$ and, by Theorem VII, p. 17, gives rise to an acceleration $+ \omega^2 \sin \varphi \cos \varphi$ about an axis orthogonal to XY and pointing toward the reader. This is the direction of $+Z$ and makes the axes right-handed; *i.e.*, rotation from X to Y, Y to Z, Z to X advances a right-handed screw along the positive directions of the axes. It is found similarly that the rotation of $\omega \cos \varphi$ at the rate $\omega \sin \varphi$ produces $- \omega^2 \cos \varphi \sin \varphi$, which means that since the two accelerations are equal but opposite, *the components of an angular velocity cannot cause acceleration by rotating each other.*

Consider a less obvious case. Evidently the resultant or instantaneous velocity suffers no vector change, because the *I.A.* has no angular velocity. But it might seem in Fig. 14 (taking n as constant in magnitude) as if the angular acceleration of the body were only $n(\theta \sin \alpha)$. There is however an *opposite* component $\theta^2 \sin \alpha \cos \alpha$ due to the turning of $\theta \sin \alpha$ by $\theta \cos \alpha$, the resultant of both being, with the value of n on page 15,

$$n\theta \sin \alpha - \theta^2 \cos \alpha \sin \alpha = \Omega\theta \sin \alpha.$$

Since $\Omega\theta \sin \alpha$ is due to the rotation of the relative velocity—the velocity relative to any frame of reference attached to the axle (not the body) and turning about the θ axis at the rate θ—we have the following result.

THEOREM VIII. *The angular velocity relative to a rotating frame is the only angular velocity whose rotation by the frame can produce angular acceleration.*

The rotating frame used in this theorem is usually a set of XYZ axes rigidly fastened to the *axle*. Generally they are not

fixed to the body; if they are, there is no relative motion and the only acceleration is that due to magnitude changes of angular velocity.

The components of θ in Fig. 14 are shown in Fig. 17, in which θ_1, for example, is the velocity with which the axle and the Y and Z axes turn about X. The resultant angular velocity ω of the body about the $I.A.$ (not shown in Fig. 17: see Fig. 14) is the vector sum of Ω (relative to XYZ) and θ_1, θ_2, θ_3.

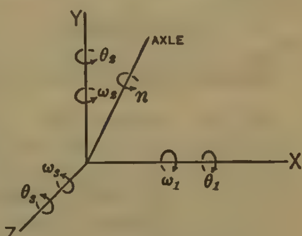


FIG. 17

The velocity of the body about the axle is n: the resultant of Ω and the projections of θ_1, θ_2, θ_3 on the axle.

The resultant or actual velocities of the body are

$$\omega_1 = \theta_1 + \Omega_1, \qquad \omega_2 = \theta_2 + \Omega_2, \qquad \omega_3 = \theta_3 + \Omega_3.$$

Hence n is the sum of the projections of ω_1, ω_2, ω_3 on the axle; Fig. 14 makes this clear.

All these relations are summed up succinctly in the *vector* equations obtained from the velocity diagram of Fig. 14:

$$\omega = \theta + \Omega = n + \theta \sin (n, \theta) \quad \textit{vectorially}.$$

Vector equations are compact but cannot be used for numerical calculation.

The angular accelerations of the axes and axle are $\dot{\theta}_1$, $\dot{\theta}_2$, $\dot{\theta}_3$. Denote the angular accelerations of the body by α, β, γ. They are found from Theorem VII, p. 17, as follows. The method is fundamental and will be used as often as the equations on p. 20 that result from it.

Scalar change of ω produces $\dot{\omega}$, whose components are $\dot{\omega}_1$, $\dot{\omega}_2$, $\dot{\omega}_3$ because ω and $\dot{\omega}$ are coaxial. Each θ rotates the two Ω's orthogonal to it; thus Ω_2 turning at the rate θ_3 produces $-\theta_3\Omega_2$ along $-X$, and Ω_3 turned by θ_2 gives $+\theta_2\Omega_3$ along $+X$; see

the convention of signs on p. 17. Hence

$$\alpha = \dot{\omega}_1 - \Omega_2\theta_3 + \Omega_3\theta_2,$$
$$\beta = \dot{\omega}_2 - \Omega_3\theta_1 + \Omega_1\theta_3,$$
$$\gamma = \dot{\omega}_3 - \Omega_1\theta_2 + \Omega_2\theta_1,$$

where

$$\dot{\omega}_1 = \dot{\theta}_1 + \dot{\Omega}_1, \qquad \dot{\omega}_2 = \dot{\theta}_2 + \dot{\Omega}_2, \qquad \dot{\omega}_3 = \dot{\theta}_3 + \dot{\Omega}_3.$$

For axes fixed to the body $\Omega = 0$.

Ex. 34. Prove $\alpha = \dot{\omega}_1 - \omega_2\theta_3 + \omega_3\theta_2$, etc.

Ex. 35. If *I.A.* in Fig. 13 is fixed, prove as in § 7 that the angular acceleration of the body about X is not the same as that of the ordinate y about X.

Ex. 36. Find the angular acceleration of the hoop in Ex. 32 about a horizontal tangent to its lowest point.

The acceleration due to the turning of the relative angular velocity, p. 18, is

$$\Omega\theta \cos\varphi = \frac{v^2}{R}\cos\varphi\left(\frac{1}{r} + \frac{\sin\varphi}{R}\right).$$

To get this from the value of γ above, the origin may be taken at any point. Let us take it at the *I.C.* of rolling, with X parallel to the axle and toward the right, Y upward through the center of the hoop, and Z toward the reader. Let the axes move so that the origin follows the point of contact, X remaining parallel to the axle and Y always passing through the center. Then

$$\begin{aligned} &\omega_1 = -n, && \omega_2 = \theta\cos\varphi, && \omega_3 = 0, \\ &\theta_1 = \theta\sin\varphi, && \theta_2 = \theta\cos\varphi, && \theta_3 = 0, \\ &\Omega_1 = \omega_1 - \theta_1, && \Omega_2 = 0, && \Omega_3 = 0. \end{aligned}$$

Hence

$$\gamma = (n + \theta\sin\varphi)\theta\cos\varphi$$

which, being positive, is from X toward Y.

11. Velocity of a Point. Consider any point $P(x, y, z)$ in Fig. 17. Its velocity relative to the axes is due only to the change of coordinates and is $\dot{x}$, $\dot{y}$, $\dot{z}$. Now if P is regarded as being on a body turning about the axle with relative velocity Ω, the rotation of this body about X gives P velocities $+ y\Omega_1$ parallel to $+ Z$, and $- z\Omega_1$ parallel to $- Y$; similarly for Ω_2, Ω_3. Since the relative rotation causes the changes in the coordinates, we have

$$\dot{x} = z\Omega_2 - y\Omega_3,$$
$$\dot{y} = x\Omega_3 - z\Omega_1,$$
$$\dot{z} = y\Omega_1 - x\Omega_2.$$

Note carefully that Ω_1, for example, is not the relative angular velocity of the *point* P; it refers to a *body* of which P is a part—even if the body consists of only the one point P; see p. 13.

The velocity of constraint, Theorem II, p. 2, produced by the rotation of the axes, which are assumed fixed at the origin, is

$$z\theta_2 - y\theta_3, \qquad x\theta_3 - z\theta_1, \qquad y\theta_1 - x\theta_2;$$

hence, by Theorem I, if the origin is at rest, the resultant velocities u, v, w of P are

$$\begin{aligned} u &= \dot{x} - y\theta_3 + z\theta_2, \\ v &= \dot{y} - z\theta_1 + x\theta_3, \quad \textit{origin at rest} \\ w &= \dot{z} - x\theta_2 + y\theta_1. \end{aligned}$$

Since any velocity u_0, v_0, w_0 of the origin is to be superposed on this, the total components are

$$u_0 + u, \qquad v_0 + v, \qquad w_0 + w.$$

Since the motion of P is due to the absolute rotation ω of the body of which it is a point, we get by reasoning as above

$$\begin{aligned} u &= z\omega_2 - y\omega_3, \\ v &= x\omega_3 - z\omega_1, \quad \textit{origin at rest} \\ w &= y\omega_1 - x\omega_2, \end{aligned}$$

where ω_1, ω_2, ω_3 are the total velocity components of the body on which P lies; they are not the angular velocities of the coordinates x, y, z; see p. 13.

Ex. 37. Derive the last equations from those preceding.
Ex. 38. Find $\ddot{x}$, $\ddot{y}$, $\ddot{z}$, $\dot{u}$, $\dot{v}$, $\dot{w}$. See p. 92.

12. Acceleration of a Point. The accelerations of $P(x, y, z)$ in Fig. 17 are got by means of Theorem III, p. 4. Along X there is $\dot{u}$ (due to increase of u), $-v\theta_3$ (due to the rotation of v at rate θ_3), $+w\theta_2$ (due to the rotation of w at rate θ_2), when the origin is at rest. Any acceleration a_0 *of the origin* is to be added to these because it increases the acceleration of constraint, p. 7.

Hence the acceleration components a, b, c of any point P are

$$a = a_0 + \dot{u} - v\theta_3 + w\theta_2,$$
$$b = b_0 + \dot{v} - w\theta_1 + u\theta_3,$$
$$c = c_0 + \dot{w} - u\theta_2 + v\theta_1,$$

where u, v, w have the values on p. 21 and are found *as if the origin were fixed.*

Ex. 39. Show that, if $\dot{u}_0 = du_0/dt$, etc.,

$$a = (\dot{u}_0 + \dot{u}) - (v_0 + v)\theta_3 + (w_0 + w)\theta_2.$$

Find and interpret the value of a_0. Compare pp. 7, 33.

Ex. 40. Find the accelerations of the top and bottom points of the hoop in Ex. 36.

The origin of the axes must be taken at some point whose acceleration is known. Therefore take axes parallel to those in Ex. 36 but with origin at the center of the hoop where $a_0 = -R\theta^2$, $b_0 = 0$, $c_0 = 0$; θ, Ω, ω have the values given.

For top point, $\dot{x} = \dot{y} = 0$, $\dot{z} = r(n + \theta \sin \varphi)$.

For bottom point, $\dot{x} = \dot{y} = 0$, $\dot{z} = -r(n + \theta \sin \varphi)$.

CHAPTER III

DYNAMICAL LAWS

13. F = ma. From his study of physics, the reader is familiar with the equation

$$F = ma,$$

in which m is the so-called "quantity of matter" of the body on which the resultant force F acts and gives the *centroid* (center of mass, center of gravity) an acceleration a in the direction and sense of F. When put into the form

$$\frac{F}{W} = \frac{a}{g} \quad \textit{where} \quad \frac{W}{g} = m,$$

it is a direct mathematical statement of Newton's First and Second Laws of Motion.

Newton's method of establishing it has been adversely criticised by Ernst Mach [1] and Karl Pearson;[2] their position gets strong support from the theory of relativity. The following article, in which Mach's point of view is adopted, is a summary of the reasoning that leads to the concepts of force and mass.

14. Mass and Force. The term *particle* denotes a physical point—a body whose rotation can be neglected. So far as translation is concerned, every body can be rigorously regarded as a physical point: see § 19.

POSTULATE OF INERTIA. ***A particle that moves with acceleration is said to be influenced.***

This is more than a verbal abbreviation, for it implies that acceleration is dependent on some circumstance that determines or conditions the motion. An uninfluenced particle moves with constant vector velocity.

[1] Mach, *Science of Mechanics*, Chap. III, §§ V–VII.

[2] Pearson, *Grammar of Science*, Chap. V.

EXPERIMENTAL PROPOSITION. *Two particles that influence each other induce contrary accelerations in each other, the numerical ratio of which is constant for the two particles and independent of the manner in which they are induced.*

Two particles influence each other when they collide, are connected by a string over a pulley (Atwood machine), connected by a string and placed on a whirling-table, etc. An experiment made in 1890 by the Hungarian physicist Eötvös [1] to test the constancy of the acceleration of gravity can be interpreted [2] as an experiment of the kind contemplated in this proposition. He found that the ratios in two different comparisons of induced accelerations differed by not more than $\pm 10^{-5}$ per cent. The constancy of the acceleration ratio for two particles therefore corresponds to an inherent property of the particles; it is independent of everything but the particles themselves and depends only—as in the following definition—on the *masses* of the particles.

DEFINITION. *If a is the acceleration induced in an arbitrarily chosen standard or unit particle P by a particle P_n, and a_n the acceleration induced in P_n by P,*

$$m_n = \frac{a}{a_n}$$

is called the mass of P_n in terms of the mass of P as unit.

DEFINITION. *The product ma of the mass m of a particle and the acceleration a induced in it is called the force F acting on the particle in the direction and sense of a; that is,*

$$F = ma.$$

As a depends on the apparatus, *e.g.*, a taut string or a stretched spring, by means or through the medium of which it is produced, we can look upon the tautness of the string or the stretch of

[1] Annalen der Physik, vol. 68, pp. 11–16.

[2] E. R. Neumann, *Relativitätstheorie* (1922), pp. 136–139.

the spring as being associated with the cause of a. That is, force can be regarded as the cause—or at any rate a concomitant—of acceleration; it is thus more than an abbreviation for ma and acquires objective reality.

15. The Pound. The unit of force used by engineers is the pound: the gravitational pull on a metal standard in vacuo at mean sea-level "at [1] any locality in which the acceleration due to gravity has the value

$$980.665 \text{ cm. per sec.}^2 = 32.1740 \text{ ft. per sec.}^2\text{"}$$

As defined here, the unit of mass is that mass for which, of course, $m = 1$ in $F = ma$. It has no name; its weight (earth pull) is $W = 1 \times g$ pounds, where g depends on the place where the weighing (force-measuring) is done.

The expressions *ten-pound mass*, *ten-pound weight*, and *ten-pound body* mean the same, *i.e.*, denote the same body, but the meaning depends on how the value *ten* was obtained. If *ten* is a spring-balance reading it is the actual earth-pull and the mass is $10/g$ units, g being the acceleration at the place of weighing. If *ten* is a lever-balance reading, the mass is 10/32.1740 units because the reading is everywhere the same as at the locality for which $g = 32.1740$. The earth-pull is $10g/32.1740$, where g refers to the place of weighing.

At the equator g is about 0.27 per cent less, and at the poles about 0.26 per cent more, than 32.1740; these differences are usually negligible in engineering and it is customary to take $g = 32.17$ for most calculations.

Ex. 41. If a body in latitude φ weighs x on a spring balance and y on a lever balance show that $xg_s = yg_\varphi$ where g_s is the standard value.

16. Weight. The earth-pull or weight mg mentioned above is not due solely to gravitational attraction. Unless otherwise stated, the latitude is zero in this article.

[1] Marks, *Mechanical Engineers' Handbook* (2d ed.), p. 73.

Let G = acceleration due to purely gravitational attraction,

g = acceleration of a freely falling body,

R = earth's radius,

ω = earth's angular velocity relative to the fixed stars,

h = height, above earth's surface, from which a particle is dropped,

r = radius vector from earth's center to particle,

θ = angle between r and a line fixed in space.

After the particle is dropped it is acted on radially by pure attraction which produces an acceleration, p. 5,

(1) $$G = r\dot{\theta}^2 - \ddot{r}.$$

Since there is no force perpendicular to r,

$$0 = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}).$$

Hence

(2) $$r^2\dot{\theta} = \text{const.} = (R + h)^2\omega = R^2\Omega,$$

where Ω is the angular velocity of the particle when it reaches the ground.

A little consideration will show that $-\ddot{r}$ (negative because r is positive outward) is what we measure in finding g, that is,

$$g = G - r\dot{\theta}^2 = G - R\omega^2, \textit{ approximately},$$

which shows that even a freely falling particle is influenced by the earth's rotation. In latitude φ,

$$g = G - R\omega^2 \cos^2 \varphi,^1$$

where, however, g and G differ slightly in direction; 11′ for $\varphi = 45°$. The following is very nearly correct:

$$g = 32.1740 - 0.085 \cos 2\varphi \textit{ ft./sec.}^2$$

A falling particle does not strike radially, for Ω is different

[1] See Routh, *Analytical Statics*, II, §§ 304–309, for a more precise treatment.

from ω: see (2). It does not strike the ground vertically below the point of release. If it is at height z at time t, (2) gives

$$\theta = \int_0^\theta d\theta = \int_0^t \left(\frac{R+h}{R+z}\right)^2 \omega dt.$$

By the simple law of falling bodies

$$h - z = \frac{1}{2}gt^2.$$

Substitution gives

$$\theta - \omega t = \frac{g\omega t^3}{3R} \text{ approximately;}$$

hence the particle strikes so many radians *eastward* of the point of release.

17. Momentum. The force F that changes the velocity of m from u to u' in time Δt is

$$F = ma = \frac{mu' - mu}{\Delta t},$$

where mu is the *linear momentum* of m. Momentum, like velocity, is a vector and $mu' - mu$ is the vector difference of momentum.

Force is therefore the vector time-rate of change of linear momentum. To find out whether this is still true when the mass is variable, let a mass element dm moving with velocity u become attached, Fig. 18, to a finite mass m moving at v. When the

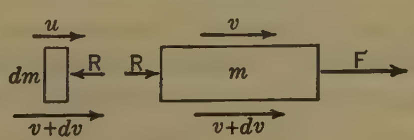


FIG. 18

two masses coalesce they will have a common velocity $v + dv$. The force R is produced at the surface of impact by the change of momentum suffered by each mass. F is the external force applied to m during the time dt of impact.

The velocity of m is increased from v to $v + dv$ by $F + R$; that of dm is decreased from u to $v + dv$ by R because R and u are opposite. Therefore

$$\textit{for } m, \qquad F + R = m\frac{dv}{dt}.$$

The force F, in $F = ma$, is in the sense of a; it must be taken as positive in the sense of the final velocity. Hence

$$\textit{for } dm, \qquad -Rdt = dm[(v + dv) - u].$$

Thus

$$Fdt = (m + dm)(v + dv) - (mv + udm),$$

which shows that force is the time-rate of change of momentum even when the mass changes through accretions from without.

In the absence of external force the vector total momentum remains unchanged by impact.

Since $dm\,dv$ vanishes in the limit, the last equation becomes

$$F = \frac{d}{dt}(mv) - u\frac{dm}{dt}.$$

According to this equation, the customary statement

$$F = \frac{d}{dt}(mv)$$

is an incomplete generalization of $F = ma$ because it omits the effect of the impact produced by the attachment of the acquired mass.

Hereafter we shall deal only with constant mass.

18. Work and Energy. The acceleration components of a particle m moving in a plane curve are $\ddot{s}$, v^2/ρ: p. 4. The tangential and normal components T, N of the force on m are therefore

$$T = m\ddot{s}, \qquad N = \frac{mv^2}{\rho}$$

from which

$$\int_1^2 Tds = \frac{mv_2^2 - mv_1^2}{2}, \qquad \frac{N_2\rho_2 - N_1\rho_1}{2} = \frac{mv_2^2 - mv_1^2}{2}.$$

The quantity $mv^2/2$ is called the *kinetic energy* E of m. The relation between kinetic energy and normal force is not generally useful, but that containing T is of fundamental significance.

$\int_1^2 Tds$ is the area under the (T, s) curve and is called the *work* done by T in the displacement from s_1 to s_2; denoting it by L, we have $Tds = dL$.

Force is thus the space-derivative d/ds of work. Work is measured in foot-pounds and is positive when force and displacement have the same sense.

Since $Tds = d(mv^2/2) = dE$, force is also the space-derivative of kinetic energy; this is true however only when E can be expressed in terms of the single variable $\dot{s}$; see Ex. 42.

Ex. 42. The force acting parallel to X on P in Fig. 7 is $F_x = ma_x$. Show that

$$F_x = \frac{d}{dt}\left(\frac{\partial E}{\partial \dot{x}}\right) - \frac{\partial E}{\partial x}$$

where, by definition, $E = \frac{m}{2}(u^2 + v^2)$; for u, v, a_x, see p. 7.

In terms of kinetic energy, we have

$$momentum = \frac{dE}{dv}.$$

Hence

$$force = \frac{d}{dt}\left(\frac{dE}{dv}\right).$$

Here v must be the *resultant* velocity and m constant; compare Ex. 42. The last relation gives

$$change\ of\ momentum = \int_1^2 (force \times dt)$$

where the time-integral of force is called *impulse*. Thus

$$force = \frac{d}{dt}(impulse).$$

The mass is to be taken as constant in all these relations.

Ex. 43. Find the infinitesimal work done by F in Fig. 18.

If m gets a displacement $ds = vdt$ in time dt we get from the value of F on p. 28

$$Fds = d\left(\frac{mv^2}{2}\right) + \frac{v^2}{2}dm - uvdm$$

which is all the work done on the entire system because the two reactions R do opposite amounts of work.

Consider now the gain of kinetic energy of both masses during impact. It is

$$\tfrac{1}{2}(m + dm)(v + dv)^2 - \tfrac{1}{2}mv^2 - \tfrac{1}{2}u^2dm = d\left(\frac{mv^2}{2}\right) - \frac{u^2}{2}dm^{1}$$

but this does not equal the work done! In fact it is less, for some of the energy produced by F has been converted into heat because in inelastic impact there is permanent deformation of the colliding bodies. As the loss evidently does not depend on F it can be calculated as if F were not acting. If V is the velocity of the combined mass after coalescence, the loss of energy is

$$\frac{u^2dm}{2} + \frac{mv^2}{2} - \frac{m + dm}{2}V^2.$$

Since F is assumed to be zero, the momentum of the system remains unchanged by impact, p. 28. It follows that

$$(m + dm)V = udm + mv;$$

hence the energy that has disappeared as kinetic is

$$\frac{dm}{2}(v^2 - 2uv + u^2).$$

If this is added to the original gain of kinetic energy of the system we get the total work done by F:

$$d\left(\frac{mv^2}{2}\right) + \frac{v^2dm}{2} - uvdm$$

which is Fds as found in the beginning.

Ex. 44. In the equation $2\int_1^2 Tvdt = m(v_2^2 - v_1^2)$, p. 28, v, v_1, v_2 are of course relative to the observer. Find under what condition the equation will be independent of the velocity of the observer.

If the observer's velocity changes by x, those in the phenomenon become $v + x$, $v_1 + x$, $v_2 + x$. The result of substituting is $\int_1^2 Tdt = m(v_2 - v_1)$; see p. 29.

Ex. 45. A mass M moving with velocity u strikes a mass m which is at rest; if the bodies are perfectly inelastic (plastic) they will move with some common velocity v after impact. Show that if the loss of kinetic energy of the system is independent of the translation of the observer, $Mu = (M + m)v$. Solved like Ex. 44.

[1] In all equations containing infinitesimals, all terms but those of the lowest order are rejected because they vanish in the limit. In the above equation, $dm\,dv$ is to be rejected. The student should keep this important rule in mind.

19. Translation. Any system, rigid or not, may be regarded as a set of particles interconnected by massless rods or by attractive or repulsive forces that are functions of the distances between the particles. The internal reactions of these constraints are pairs of equal, opposite forces.

Let F and R be the projections, on any line, of the resultant external forces and internal reactions acting on *one* particle m; call a the corresponding projection of the acceleration of m.[1] Then

$$F + R = ma,$$

and for all the particles

$$\Sigma(F + R) = \Sigma ma.$$

Since the R's occur in equal opposite pairs and cancel, we have

$$\Sigma R = 0,$$

and as we can always take a fixed X-axis parallel to a,

$$a = \ddot{x}.$$

Hence

$$\Sigma F = \Sigma m\ddot{x}.$$

If x is the abscissa of the one particle m, we can always find an x_c to satisfy the relation

$$\Sigma mx = x_c \Sigma m;$$

hence, putting $\Sigma m = M$, we find

$$\Sigma F = \Sigma m\ddot{x} = M\ddot{x}_c,$$

which is true for any, *i.e.*, every, direction of projection. *The value of x_c defines the abscissa of a point called the* **centroid** *of the system.*

The last equation may be interpreted as follows.

THEOREM IX. ***The motion of the centroid of any system is independent of the position of the applied forces.***

[1] The student should draw the diagram.

Ex. 46. Two particles weighing 1 and 3 lbs. are 12 ft. apart. Verify by computation that their centroid will be shifted the same distance whether a force of 5 lbs. is applied for 2 secs. in a given direction to one mass or to the other.

This illustrates an uncommon case of Theorem IX.

Ex. 47. Show that collisions among the parts of a system cannot change its momentum.

Put $\Sigma F = M\ddot{x}_c = 0$.

20. General Equations of Rotation. For any particle m referred to a set of moving axes in space the equations of translation parallel to Y and Z are, p. 31,

$$F_2 + R_2 = mb, \qquad F_3 + R_3 = mc,$$

where b, c are the accelerations parallel to Y, Z.

The sum of the *moments* about X of the forces on all the particles of a system are

$$L = \Sigma(F_3 y - F_2 z) = \Sigma m(cy - bz)$$

in which the moments of the internal reactions have cancelled out. The sense of a moment follows the convention of signs on p. 17.

From this and the two similar equations for the moments M, N about Y, Z, it is easy to eliminate a, b, c, u, v, w by means of the relations on pp. 21, 22; the results are

$$(1) \qquad \begin{cases} L = L_0 + \dot{h}_1 - h_2\theta_3 + h_3\theta_2, \\ M = M_0 + \dot{h}_2 - h_3\theta_1 + h_1\theta_3, \\ N = N_0 + \dot{h}_3 - h_1\theta_2 + h_2\theta_1, \end{cases}$$

where

$$(2) \qquad \begin{cases} L_0 = \Sigma m(c_0 y - b_0 z), \\ M_0 = \Sigma m(a_0 z - c_0 x), \\ N_0 = \Sigma m(b_0 x - a_0 y), \end{cases}$$

and

$$(3) \qquad \begin{cases} h_1 = \Sigma m\{(y^2 + z^2)\omega_1 - xy\omega_2 - zx\omega_3\}, \\ h_2 = \Sigma m\{(z^2 + x^2)\omega_2 - yz\omega_3 - xy\omega_1\}, \\ h_3 = \Sigma m\{(x^2 + y^2)\omega_3 - zx\omega_1 - yz\omega_2\}. \end{cases}$$

The quantities h_1, h_2, h_3 are called the components of the *angular momentum* h of the set of particles. Angular momentum is a vector because *it is defined as having components.*

Ex. 48. Prove $h_1 = \Sigma m(wy - vz)$.

This shows that angular momentum is the moment of linear momentum; for this reason it is often called *moment of momentum.*

Ex. 49. Using the values of α, β, γ of Ex 34, p. 20, and $\dot{x}$, $\dot{y}$, $\dot{z}$ from § 11, prove

$$L = L_0 + \Sigma m(y^2 + z^2)\alpha + \Sigma m\{(y^2 + x^2) - (x^2 + z^2)\}\omega_2\omega_3$$
$$- \Sigma mxy(\beta - \omega_1\omega_3) - \Sigma mzx(\gamma + \omega_2\omega_1) + \Sigma myz(\omega_3^2 - \omega_2^2).$$

If we substitute into (2) the coordinates ξ, η, ζ of the centroid of the system, where, for instance, $\xi\Sigma m = \Sigma mx$, we get the important relations

$$(4) \qquad \begin{cases} L_0 = (c_0\eta - b_0\zeta)\Sigma m, \\ M_0 = (a_0\zeta - c_0\xi)\Sigma m, \\ N_0 = (b_0\xi - a_0\eta)\Sigma m, \end{cases}$$

where a_0, b_0, c_0 are the accelerations of the origin; see Ex. 39.

If the vectors $a_0\Sigma m$, $b_0\Sigma m$, $c_0\Sigma m$ are drawn at the *centroid,* then L_0, M_0, N_0 are the moments of these vectors about the axes.

L_0, M_0, N_0 vanish when the origin is fixed, when it is at the centroid, or when the acceleration of the origin (the resultant of a_0, b_0, c_0) passes through the centroid.

Equation (1) resembles that for a, b, c, p. 22, and expresses a theorem in some respects like Theorem III, p. 4, and Theorem VII, p. 17.

THEOREM X. ***Moment of force produces change of angular momentum. The moments, about any axis, of the external forces on a system equal the resultant of***

(i) ***the moment of the mass $\times$ acceleration of the origin, the fictitious force vector being drawn at the centroid,***

(ii) ***the time-rate, d/dt, of the angular momentum about the axis of moments,***

(iii) ***the product of the angular momentum about a perpendicular axis by the angular velocity of that axis. This product has the direction in which the head of the angular momentum vector turns.*** See Theorem VII, p. 17.

That Theorem X is the interpretation of (1) is seen thus. Consider the equation for L: L_0 and $\dot{h}_1$ are evidently the quantities stated in (i), (ii); for part (iii) observe that the Y axis (perpendicular to the axis of L) turns about Z at the rate θ_3, carries with it the angular momentum vector h_2, and thereby produces a vector change-rate $-h_2\theta_3$ which is negative because it is directed along $-X$; similarly the vector h_3 turns at θ_2 and produces $+h_3\theta_2$ along $+X$.

21. Plane Rotation of a Rigid Body. Before discussing the general equations (1), p. 32, we shall study the simpler case of rotation about an axis that does not change its direction. If Z is the axis of constant direction the XY axes may turn about it at speed $\theta_3 = \theta$, but $\theta_1 = \theta_2 = 0$. Likewise the body may turn about Z at speed $\omega_3 = \omega$, but not about X or Y, *i.e.*, $\omega_1 = \omega_2 = 0$. Equations (1), (3), and (4), above, give

$$L = L_0 - \frac{d}{dt}\Sigma mxz\omega + \theta\omega\Sigma myz,$$

$$M = M_0 - \frac{d}{dt}\Sigma myz\omega - \theta\omega\Sigma mzx,$$

$$N = N_0 + \frac{d}{dt}\Sigma m(x^2 + y^2)\omega.$$

Observe that even for plane rotation L and M are not necessarily zero. Cases in which they are not zero are discussed in § 25.

Certain special cases of the equation for N are interesting.

(i) *Origin fixed in space and on the body.* $N_0 = 0$; see the interpretation of (4), p. 33. As the distance of any one particle from the axis is constant, $x^2 + y^2 = r^2$ is constant. Evidently ω is common to all the particles; therefore, putting $\Sigma mr^2 = I$

for brevity, we get

$$N = I\dot{\omega} = I\alpha,$$

where I is called the *moment of inertia* of the body about Z. This is the familiar equation of elementary mechanics.

(ii) *Origin fixed on the body but not in space.* Here

$$N = N_0 + I\dot{\omega},$$

which is slightly more general than the result in (i). Note that $N_0 = 0$ when the origin is at the centroid or when the acceleration of the origin passes through the centroid.

Ex. 50. Prove

$$N = I_c\dot{\omega} + (\textit{moment, about origin, of } a_c\Sigma m),$$

where I_c is the moment of inertia about, and A_c the acceleration of, the centroid; N as in case (ii).

Use the theorem $I = I_c + l^2\Sigma m$, Ex. 64, p. 40, where l is the distance between the parallel axes about which I, I_c are taken, *i.e.*, l is the distance between origin and centroid.

(iii) *Origin fixed in space but not on the body.* $N_0 = 0$; ω may be expressed in terms of the angular velocity ω_c of the radius vector from origin to centroid, and the angular velocity ω_r of the body relative to this line:

$$\omega = \omega_c + \omega_r.$$

It follows that

$$N = \frac{d}{dt}(I\omega_c) + \frac{d}{dt}(I\omega_r),$$

where I refers to the origin.

Ex. 51. Show that in (iii)

$$N = \Sigma mra_p,$$

where a_p has the value given in Ex. 17, p. 8.

(iv) *Origin at the instantaneous center.* It is left for the reader to prove that

$$N = I\dot{\omega} + \frac{\omega}{2}\frac{dI}{dt}.$$

The result follows from Ex. 51, in which a_p is given on p. 9.

Ex. 52. The kinetic energy of a system rotating about an instantaneous center is $\frac{1}{2}\Sigma mr^2\omega^2 = \frac{1}{2}I\omega^2$. Equate the change of kinetic energy, $d(\frac{1}{2}I\omega^2)$, to the work, $N\omega dt$, of the external forces and obtain the result in (iv).

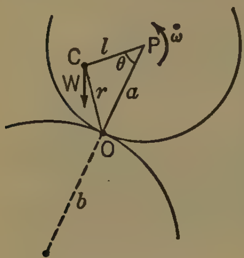


FIG. 19

Ex. 53. A non-homogeneous sphere, radius a, center P, centroid C, mass m, rolls without slipping on a fixed sphere of radius b, Fig. 19. Form the equation of rotation.

Let ω be the absolute angular velocity of the rolling sphere and $\dot{\theta}$ the relative velocity of rolling. By p. 11 the acceleration of O is $a\omega\dot{\theta}$ directed from O toward P. Now by means of Theorem IV, p. 7, find the acceleration of C from that of O. Substitution in the equation of Ex. 50 gives

$$N = I_c\dot{\omega} + m(a\omega\dot{\theta}l \sin \theta + r^2\dot{\omega})$$
$$= I\dot{\omega} + ma\omega\dot{\theta}l \sin \theta$$

because $I = I_c + mr^2$ from Ex. 64, p. 40.

Observe that the second form of N is that of case (ii). To use case (iv) we have

$$I = I_c + mr^2, \quad r^2 = l^2 + a^2 - 2la \cos \theta.$$

Therefore, case (iv),

$$N = I\dot{\omega} + \frac{\omega}{2}\frac{d}{dt}(I_c + mr^2)$$
$$= I\dot{\omega} + ma\omega\dot{\theta}l \sin \theta.$$

Ex. 54. Prove

$$N = -W\{l \sin (\theta + \varphi) - a \sin \varphi\},$$

where W is the weight of the sphere in Fig. 19 and φ is the angle OP makes with the vertical. N is negative because it is opposite to the clockwise direction assumed in Ex. 53.

Ex. 55. Show that the cyclic (periodic) time of a small oscillation of a hemisphere on a rough plane (no slip) is

$$2\pi\left(\frac{I_c + m(a - l)^2}{mgl}\right)^{1/2},$$

where the symbols have the meaning in Fig. 19.

Equate N from Exs. 53, 54 and neglect $\omega\dot{\theta}$ because the oscillations are small. The resulting equation will be like that of simple harmonic motion.

Ex. 56. Show that if the plane in Ex. 55 is smooth the time is

$$2\pi\left(\frac{I_c}{mgl}\right)^{1/2}.$$

If the plane is smooth the hemisphere will slip but its centroid will not move *horizontally.*

Ex. 57. A rod of length r lies in a smooth vertical circle of radius R. Find the time of a small oscillation. Check the result by taking the origin first at the center of the rod and then at the center of the circle.

22. Rigid Body in Space. When the system is rigid, all particles have a common angular acceleration and a common angular velocity, provided the origin is fixed on the body. This is the most useful case and it is assumed here.

For brevity put in (3), p. 32,

$$(1)\quad \begin{cases} A = \Sigma m(y^2 + z^2), & B = \Sigma m(z^2 + x^2), \quad C = \Sigma m(x^2 + y^2), \\ D = \Sigma myz, & E = \Sigma mzx, \qquad\qquad F = \Sigma mxy. \end{cases}$$

A, B, C are called the *moments of inertia* of the body about the axes X, Y, Z; D, E, F are called *products of inertia*. These names are obviously not descriptive and serve merely as labels.

With this notation (3), p. 32, become

$$(2)\qquad \begin{cases} h_1 = A\omega_1 - F\omega_2 - E\omega_3, \\ h_2 = B\omega_2 - D\omega_3 - F\omega_1, \\ h_3 = C\omega_3 - E\omega_1 - D\omega_2, \end{cases}$$

which are to be used in (1), p. 32, *when the body is rigid with the origin fixed on it.*

When the axes are not fixed to the rigid body, A, B, C, D, E, F are variable except under special circumstances; this makes (1), p. 32, inconvenient to use. In some cases D, E, F are zero but they may nevertheless be variable as in Ex. 60, p. 38; A, B, C cannot be zero.

When $A, \cdots, F$ are variable the equation in Ex. 49, p. 33, is free from their derivatives and is preferable to the equations containing the angular momenta. We have then

$$(3)\qquad \begin{aligned} L = L_0 + A\alpha - (B - C)\omega_3\omega_2 + D(\omega_3{}^2 - \omega_2{}^2) \\ - E(\gamma + \omega_2\omega_1) - F(\beta - \omega_3\omega_1); \end{aligned}$$

M and N are obtained by cyclic permutation of the symbols. For α, β, γ, see p. 20.

These formidable equations (3) can be much simplified by a proper choice of axes. It is shown on p. 39 that at any point in a rigid body there are always three orthogonal axes for which

D, E, F vanish; if they are taken as fixed to the body, $\alpha = \dot{\omega}_1$, $\beta = \dot{\omega}_2$, $\gamma = \dot{\omega}_3$ and (3) reduce to Euler's equations,[1] discovered in 1736:

$$\text{(4)} \qquad \begin{cases} L = L_0 + A\dot{\omega}_1 - (B - C)\omega_2\omega_3, \\ M = M_0 + B\dot{\omega}_2 - (C - A)\omega_3\omega_1, \\ N = N_0 + C\dot{\omega}_3 - (A - B)\omega_1\omega_2. \end{cases}$$

For L_0, M_0, N_0 see (4), p. 33. Compare also the interpretation of Theorem X, p. 33.

Ex. 58. Show from (4) that when $L_0 = M_0 = N_0 = 0$ and $\dot{\omega} = 0$ the resultant of L, M, N is normal to ω.

This result proves that the last terms of (4) represent moments that tend to change the orientation of the ω axis. Or from the opposite point of view they arise from the change of direction of ω; see § 26.

Ex. 59. Derive (4) directly from Theorem X, p. 33.

Ex. 60. Prove $\dfrac{dD}{dt} = (C - B)\Omega_1$ when $D = 0$, $E = 0$, $F = 0$.

Differentiate Σmyz and use the equations in § 11.

Ex. 61. Using α from p. 20, verify that (3) becomes

$$\text{(5)} \qquad \begin{aligned} L = L_0 + A\dot{\omega}_1 - (A + B - C)\Omega_2\theta_3 + (A - B + C)\Omega_3\theta_2 \\ + (C - B)(\theta_2\theta_3 + \Omega_2\Omega_3) + D(\omega_3^2 - \omega_2^2) - E(\gamma + \omega_2\omega_1) \\ - F(\beta - \omega_3\omega_1). \end{aligned}$$

23. Angular Momentum; Moment of Inertia. Angular momentum h is defined as the resultant of the vectors h_1, h_2, h_3, pp. 32, 37. The form of h_1, for example, shows that a body may have angular momentum about an axis X around which there is no angular velocity ω_1. The axes of h and ω do not coincide, because h_1, h_2, h_3 are in general not proportional to ω_1, ω_2, ω_3. If φ is the angle between ω and h, we have

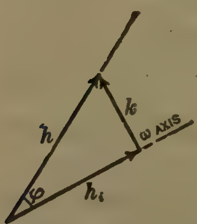


FIG. 20

$$\begin{aligned} h\omega \cos \varphi &= h_1\omega_1 + h_2\omega_2 + h_3\omega_3 \\ &= A\omega_1^2 + B\omega_2^2 + C\omega_3^2 \\ &\quad - 2D\omega_2\omega_3 - 2E\omega_3\omega_1 - 2F\omega_1\omega_2. \end{aligned}$$

In Fig. 20, $h \cos \varphi = h_i$ is the angular momentum about the ω or instantaneous axis; k is the angular momentum about an

[1] Euler discussed only the special case of $L_0 = M_0 = N_0 = 0$.

axis normal to ω, although the body has no velocity about the k axis. If the direction cosines[1] of ω are λ, μ, ν

$$\omega_1 = \lambda\omega, \qquad \omega_2 = \mu\omega, \qquad \omega_3 = \nu\omega,$$

and

$$(1) \qquad \begin{aligned} h_i &= \omega(A\lambda^2 + B\mu^2 + C\nu^2 - 2D\mu\nu - 2E\nu\lambda - 2F\lambda\mu) \\ &= \omega I_i, \end{aligned}$$

where I_i stands for the parenthesis. When the instantaneous axis coincides with X, Y, Z, the quantity I_i takes the values A, B, C respectively; I_i is called the *moment of inertia of the body about the I.A.* Thus, for any line through the origin, direction cosines l, m, n, the moment of inertia is

$$(2) \qquad I = Al^2 + Bm^2 + Cn^2 - 2Dmn - 2Enl - 2Flm.$$

Although I depends on the orientation of its axis, it is not a vector, because it has no direction (sense) *to or fro along* the axis. Putting

$$l^2 = I\xi^2, \qquad m^2 = I\eta^2, \qquad n^2 = I\zeta^2$$

in (2), we obtain

$$(3) \qquad A\xi^2 + B\eta^2 + C\zeta^2 - 2D\eta\zeta - 2E\zeta\xi - 2F\xi\eta = 1,$$

which represents a quadric surface in terms of the variables ξ, η, ζ taken as coordinates.

Since

$$1 = l^2 + m^2 + n^2 = I(\xi^2 + \eta^2 + \zeta^2),$$

the radius vector to any point (ξ, η, ζ) on the surface is $I^{-1/2}$. As this is real and finite, Ex. 62, p. 40, the surface is an ellipsoid, and is called the *inertia ellipsoid.*

The coordinate axes in (3) can always be rotated to coincide with the axes of the ellipsoid. This makes the *product* terms—those containing $\xi\eta$, etc.—vanish; (3) then takes the simpler form

$$(4) \qquad A\xi^2 + B\eta^2 + C\zeta^2 = 1,$$

whence

$$(5) \qquad I = Al^2 + Bm^2 + Cn^2,$$

[1] λ = cosine of angle between ω, ω_1; $\lambda^2 + \mu^2 + \nu^2 = 1$.

where A, B, C, ξ, η, ζ, l, m, n refer to new axes and are different in value from the letters used in (2), (3).

A, B, C in (4), (5) are called *principal* moments of inertia and their axes are the *principal axes.*

Since D, E, F do not occur in (4), (5) *the products of inertia vanish for principal axes.*

The principal moments of inertia are greatest and least because the axes of the inertia ellipsoid are greatest and least.

If A, B, C are in decreasing order of magnitude their axes—the radius vectors $A^{-1/2}$, $B^{-1/2}$, $C^{-1/2}$ to the ellipsoid (4)*—are in increasing order of magnitude.*

Ex. 62. Prove that I in (2), like A, B, C, (1), p. 37, is Σmr^2 where r is the perpendicular from m to the axis.

The direction cosines of the radius vector R from the origin to any point (x, y, z) are x/R, y/R, z/R. Hence if φ is the angle between the axis (l, m, n) and R,

$$R \cos \varphi = lx + my + nz;$$

$$\therefore \quad r^2 = R^2 \sin^2 \varphi = x^2 + y^2 + z^2 - (lx + my + nz)^2.$$

But $1 - l^2 = m^2 + n^2$, $1 - m^2 = l^2 + n^2$, $1 - n^2 = l^2 + m^2$; whence the required result, which shows, by the way, that I cannot be negative.

Ex. 63. In Fig. 20 prove $k^2 = a\alpha^2 + b\beta^2 + c\gamma^2 - 2d\beta\gamma - 2e\gamma\alpha - 2f\alpha\beta$, where $a = h_2^2 + h_3^2, \cdot \cdot \cdot, d = h_2h_3, \cdot \cdot \cdot,$ and α, β, γ are the direction cosines of h.

Ex. 64. Prove $I = I_c + md^2$, where I refers to any axis and I_c to a parallel axis through the centroid; d is the distance between the axes and m is the mass of the body.

Look parallel to the axes and draw a diagram in which they appear as points at C (centroid) and O (other axis). If r, ρ are the respective distances from any m to C, O

$$I = \Sigma m\rho^2, \quad I_c = \Sigma mr^2.$$

Now study the relation between r^2, ρ^2.

Ex. 65. For a rigid body turning about a fixed point the velocity of any particle m is $r\omega$, where r is the distance from m to the resultant or instantaneous ω axis. Prove that E, the kinetic energy of rotation, is given by

$$\begin{aligned} 2E &= I_i\omega^2 = h\omega \cos \varphi \\ &= A\omega_1^2 + B\omega_2^2 + C\omega_3^2 - 2D\omega_2\omega_3 - 2E\omega_3\omega_1 - 2F\omega_1\omega_2. \end{aligned}$$

Ex. 66. Prove $G_i\omega dt = dE$ and therefore

$$G_i = \frac{d}{dt}(I_i\omega) - \frac{\omega}{2}\frac{dI_i}{dt},$$

where E is the kinetic energy, Ex. 65, and G_i is the sum of the moments of the applied forces about the instantaneous axis; cf. Ex. 52. Note that since $I_i^{-1/2}$ is the radius vector of an ellipsoid, dI_i is not zero unless the axis is fixed in the body or unless it happens to be passing through a principal axis. In the latter case the radius vector to the inertia ellipsoid passes through a maximum value.

Ex. 67. Prove that the necessary and sufficient conditions for X to be a principal axis are $\Sigma myx = 0$, $\Sigma mzx = 0$.

If X is principal it is an axis of the ellipsoid (3), p. 39, using x, y, z in place of ξ, η, ζ. Hence a tangent plane at $y = 0$, $z = 0$ is parallel to YZ. Or if $y = 0$, (3) defines an ellipse for which $dx/dz = 0$ at $z = 0$ in order that X may be principal; also if $z = 0$, $dx/dy = 0$ at $y = 0$.

Ex. 68. Show from (4), p. 38, that when the instantaneous axis lies along a principal axis there will be no moments to change the direction of rotation; see Ex. 58, p. 38.

Ex. 69. Rectangular axes ξ, η, Z are turned about Z through an angle φ with respect to principal axes X, Y, Z. Find $\Sigma m\xi\eta$ in terms of x, y, z.

$$\xi = x \cos \varphi - y \sin \varphi, \quad \eta = x \sin \varphi + y \cos \varphi.$$

Hence

$$\begin{aligned}\Sigma m\xi\eta &= \Sigma m(x^2 - y^2) \sin \varphi \cos \varphi \\ &= \Sigma m\{(x^2 + z^2) - (y^2 + z^2)\} \sin \varphi \cos \varphi \\ &= (B - A) \sin \varphi \cos \varphi.\end{aligned}$$

Ex. 70. Prove as in Ex. 69 that if the direction cosines of rectangular axes ξ, η with respect to principal axes are (α, β, γ) for ξ and (λ, μ, ν) for η,

$$\Sigma m\xi\eta = -A\alpha\lambda - B\beta\mu - C\gamma\nu,$$

where

$$\alpha\lambda + \beta\mu + \gamma\nu = 0.$$

CHAPTER IV

SIMPLE GYROSCOPIC PHENOMENA

24. Gyroscopic Phenomena. Any change in the direction of the angular momentum vector is accompanied by torque or moment of force: Theorem X, p. 33. These directional changes are usually ignored in elementary mechanics, and the gyroscopic phenomena arising from them become needlessly astonishing to the student. They may be present in simple cases of steady motion, as, for example, in the conical pendulum, which is no less—and no more—successful than a spinning top in "defying the law of gravitation."

The top, either the play top or the flywheel with its axle mounted in Cardan rings,[1] is the classic example of the gyroscopic effect. It spins with its axis out of the vertical, rises apparently unaided, resists any attempt to make it fall, and seems never to respond properly to any stimulus. The behavior of the top may be strange, but that of the angular momentum vector is all that it should be.

This chapter deals only with the steady motion of rotating bodies. In steady motion, which corresponds to equilibrium in statics, there are no magnitude changes of angular velocity; consequently the dynamical equations of motion lose their differential character and become algebraic force relations.

25. Rotating Balance. A rotating system that requires reactions at the bearings of its axle to equilibrate the centripetal forces and to maintain the position of the axis of rotation is said to be unbalanced or out of balance. The problem of determining the bearing reactions, or of finding what masses should be added to the system to balance it, can be solved by means of the condi-

[1] See Fig. 51.

tions of statical equilibrium. It will now be shown that a real statical problem such as this may contain the ideas discussed in Chapter III.

Two massless coplanar arms are mounted on a shaft running at n radians per second. Masses m_1, m_2 are concentrated at their ends as in Fig. 21.

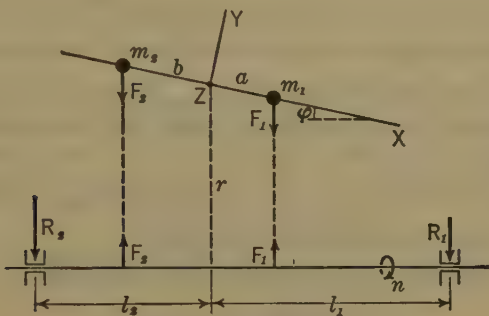


FIG. 21

The sum of the forces F_1, F_2 exerted by the arms on the masses and also on the shaft is

$$\begin{aligned} F_1 + F_2 &= m_1(r - a \sin \varphi)n^2 + m_2(r + b \sin \varphi)n^2 \\ &= R_1 + R_2 \end{aligned}$$

because F_1 and F_2 accelerate the masses but balance the bearing reactions; the weights m_1g, m_2g of the masses are omitted since it is desired to study only the effect of rotation. The rotating system is said to be *out of balance* unless R_1 and R_2 are zero.

Consider the effect of the moments of the *downward* F_1, F_2 about Z (normal to the paper):

$$\begin{aligned} F_2b \cos \varphi - F_1a \cos \varphi &= m_2(r + b \sin \varphi)n^2b \cos \varphi \\ &\qquad - m_1(r - a \sin \varphi)n^2a \cos \varphi \\ &= R_2l_2 - R_1l_1 \\ &= N, \textit{ say.} \end{aligned}$$

Hence

(1) $N = -(m_1a - m_2b)rn^2 \cos \varphi + (m_1a^2 + m_2b^2)n^2 \sin \varphi \cos \varphi.$

If x_c is the abscissa of the centroid of m_1 and m_2, we have

$$(m_1 + m_2)x_c = m_1a - m_2b;$$

hence

$$\begin{aligned} -(m_1a - m_2b)rn^2 \cos \varphi &= -(m_1 + m_2)(rn^2)(x_c \cos \varphi) \\ &= N_0, \textit{ say,} \end{aligned}$$

where N_0, p. 33, is the moment, about Z, of the product of mass $(m_1 + m_2)$ by the acceleration rn^2 of the origin, the lever arm of the moment being $x_c \cos \varphi$. N_0 is negative because it acts left-handed, *i.e.*, from Y to X; see the convention of signs given on p. 17.

The last term in (1) has two interesting interpretations depending on the orientation of the axes.

For XYZ as in Fig. 21, moving with the system,

$$n \sin \varphi = \omega_2$$

is the angular velocity of the two-mass system about Y;

$$m_1a^2 + m_2b^2 = B$$

is the moment of inertia about Y; and

$$n \cos \varphi = \omega_1$$

is the rate at which the angular momentum vector $B\omega_2$ is turning about X toward $+Z$. (See Theorem X, p. 33.)

Hence (1) becomes

$$N = N_0 + B\omega_2\omega_1, \tag{2}$$

which is the same solution as that obtained from the third Euler equation, (4), p. 38, because the axes are fixed to the rotating masses *and are also principal* since

$$D = E = F = 0.$$

If the axes are turned about Z until X is parallel to the shaft, we may put

$$a \cos \varphi = x_1, \quad a \sin \varphi = -y_1, \quad b \cos \varphi = -x_2, \quad b \sin \varphi = y_2.$$

Equation (1) becomes

$$
\begin{aligned}
(3) \qquad N &= -(m_1x_1 + m_2x_2)rn^2 - (m_1x_1y_1 + m_2x_2y_2)n^2 \\
&= N_0 - Fn^2,
\end{aligned}
$$

where N_0 is interpreted as before, and F is the product of inertia; *these axes are not principal axes.*

Ex. 71. Get (3) from the result in Ex. 49, p. 33.
Ex. 72. Get (3) from (1), p. 32.
Ex. 73. Find under what conditions any rigid system of rotating masses will be balanced.

For any set of masses arranged around the shaft, the forces exerted on the shaft can be resolved into components parallel to two orthogonal planes XY, XZ where X lies along the shaft. The forces are of the type myn^2, mzn^2; two necessary, but not *sufficient*, conditions for no bearing reactions are

$$n^2\Sigma my = 0, \quad n^2\Sigma mz = 0,$$

or the centroid must lie on the shaft.

In order that there may be no moments about Y, Z

$$\Sigma myn^2x = 0, \quad \Sigma mzn^2x = 0$$

or

$$F = 0, \quad E = 0.$$

The shaft must therefore be a principal axis through the centroid. It is shown in Chapter V that when a body under no external forces rotates about a principal centroidal axis, no forces are required to maintain the position of the axis of rotation.

The above four equations may be solved for four unknown; as the unknowns may always be of the type m_1y_1, m_1z_1, m_2y_2, m_2z_2, it follows that any rotating system can be balanced by two suitably placed masses.

26. Use of the General Equations. To illustrate the use of Theorem X, p. 33, and the general equations (1)–(4) in § 22, we shall deal with a rotating symmetric disk keyed on a shaft through its centroid but not perpendicular to the disk; the shaft runs at n radians per second.

Take *principal* axes *fixed to the disk:* with X normal to it, and the XY plane vertical in the position illustrated. A, B, B are the principal moments of inertia.

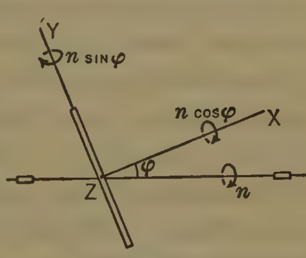


FIG. 22

By Theorem X, the angular momentum $An \cos \varphi$ about X turns toward $+ Z$ at the rate $n \sin \varphi$, and therefore requires a moment $An^2 \sin \varphi \cos \varphi$ about $+ Z$ (*i.e.*, from X to Y) to maintain the rotation of the angular momentum vector. This is like the inward radial force required to maintain the inward radial acceleration of a particle moving in a circle.

Since the senses of vectors denoting or associated with rotation are of the utmost importance because they are easily got wrong, it will not be superfluous to state again that all such vectors are to be pointed in the direction along which a right-handed screw would be advanced by the rotation. Any vector is positive when it points in the positive sense of an axis.

The arrow of the vector $Bn \sin \varphi$ points along $- Y$ because $n \sin \varphi$ rotates X to Z, *i.e.*, a right-handed screw would advance along $- Y$ when turned from X to Z. Now $n \cos \varphi$ turns the arrow of $Bn \sin \varphi$ toward $- Z$ and thus requires a right-handed moment $Bn^2 \sin \varphi \cos \varphi$ about $- Z$, and therefore a left-handed moment $- Bn^2 \sin \varphi \cos \varphi$ about $+ Z$: see the convention of signs, p. 17, and Theorem X, (iii).

The total moment due to deviations of angular momenta, but not including that due to scalar changes, is thus

$$N = (A - B)n^2 \sin \varphi \cos \varphi.$$

As the velocities are constant and the disk has no motion relative to the axes, N is the entire moment. This is the result given by Euler's equation, p. 38, for $N_0 = 0$, $\dot{\omega}_3 = 0$.

If $A > B$, which is usually the case since the disk is a cylinder or prism when $A < B$, the moment is positive, *i.e.*, from X to Y, and is therefore produced by reactions at the upper left and lower right bearings. The disk will evidently tend to set itself normal to the shaft. Since N is independent of the positions of the bearings, it is a couple, the bearing reactions being equal and opposite. This follows also from the fact that the centroid lies on the shaft.

Ex. 74. Find the ratio of thickness to diameter of a circular disk so that the bearing reactions in Fig. 22 will vanish.

It is instructive to derive the value of N by using axes that are *not principal*. Take, for example, rectangular axes ξ, η, Z fixed to the disk, ξ being horizontal and η momentarily vertical in Fig. 22. Then in (1), (3), p. 32,

$$\theta_1 = n, \qquad \theta_2 = \theta_3 = 0, \qquad \omega_1 = n, \qquad \omega_2 = \omega_3 = 0,$$
$$h_1 = \Sigma m(\eta^2 + z^2)n, \qquad h_2 = -\Sigma m\xi\eta n, \qquad h_3 = -\Sigma m\xi z n.$$

Here $h_3 = 0$ because, on account of symmetry about the $\xi\eta$ plane, for every $+m\xi z$ there is an equal $-m\xi z$. Hence

$$\begin{aligned}\Sigma m(\eta^2 + z^2) &= \textit{moment of inertia about } \xi \\ &= A\cos^2\varphi + B\sin^2\varphi,\end{aligned}$$

by (2), p. 39, where A, B refer to X, Y in Fig. 22.

From Ex. 69, p. 69,

$$\Sigma m\xi\eta = (B - A)\sin\varphi\cos\varphi.$$

Now $N_0 = 0$ either because the origin is at rest or because the centroid is at the origin. Since there is no motion relative to the axes, $\dot{h}_3 = 0$: *i.e.*, $\dot{\xi} = \dot{\eta} = \dot{z} = 0$ by § 11. Observe that $h_3 = 0$ does not necessitate $\dot{h}_3 = 0$. The equation for N, p. 38, thus reduces to that found above:

$$N = -(B - A)n^2\sin\varphi\cos\varphi.$$

Ex. 75. Find N from Ex. 61, p. 38.

27. The Gyroscopic Moment. The typical gyroscopic effect occurs as a result of turning the axle of a rotating body. The body exerts a moment about an axis normal to the axle and to the line about which the axle is turned. The problem of finding the moment is not as simple as it looks and is often solved by incorrect methods.

In Fig. 23 a horizontal axle carrying an unsymmetric disk is turned at the rate θ about a vertical line. As in § 8, the velocity ω about the instantaneous axis is the resultant of the

spin n about the axle and the velocity θ of the axle. The diagram bears a superficial and deceptive resemblance to Fig. 22. That is, the disk, rotating about an axis $I.A.$ not normal to it, *seems* as in Fig. 22 to exert only a moment $+ (A - B)n\theta$ on the bearings. The great difference between the two cases is that the $I.A.$ in Fig. 22 is fixed in space, whereas in Fig. 23 it turns about Y.

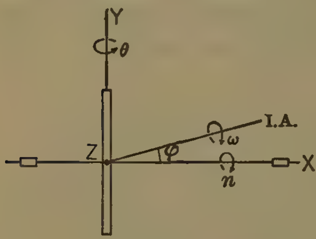


FIG. 23

Let us use Ex. 61, p. 38, taking Y as a fixed vertical. At the instant the principal axes coincide with the axes XYZ, *fixed in space*, the products of inertia with respect to XYZ vanish. If this is the case,

$$\theta_1 = 0, \quad \theta_2 = \theta, \quad \theta_3 = 0, \quad \omega_1 = n, \quad \omega_2 = \theta, \quad \omega_3 = 0,$$
$$\Omega_1 = n, \qquad \Omega_2 = \Omega_3 = 0, \qquad N_0 = 0.$$

Here $\dot{\omega}_3 = 0$ because ω_3 is independent of any position into which θ_2 can move Z; this is not so in all problems, as is shown below. With these values Ex. 61 gives

$$(1) \qquad N = -(C + A - B)n\theta.$$

We have just used Y as fixed in space; assume XYZ now to be moving with the body and to be principal. From (4), p. 38,

$$N = C\dot{\omega}_3 - (A - B)n\theta.$$

Here $\dot{\omega}_3$ is not zero, although $\omega_3 = 0$, because ω_3 depends on the momentary position of Z. Thus when Z is ψ below the horizontal,

$$\omega_3 = -\theta \sin \psi,$$

where θ is the given constant velocity about the vertical: not about Y. Since $\dot{\psi} = n$, we have

$$\dot{\omega}_3 = -n\theta \quad \text{for} \quad \psi = 0,$$

and

$$(2) \qquad N = -Cn\theta - (A - B)n\theta.$$

Therefore the moment about a principal axis is the same whether the axis of the moment is fixed in space or fixed in the body.

If there are no constraints to supply the moment N there will be angular velocity about Z and the motion will not be steady, *i.e.*, the axle will not remain horizontal. N produces no scalar change of velocity but it may be looked upon as the cause of the angular acceleration $n\theta$.

Ex. 76. Taking Y as fixed vertically, show that no moment M is needed to maintain the velocity θ.

When the rotor in Fig. 23 is a solid of revolution, $B = C$ and $N = -An\theta$, which is also approximately true for a wheel with many spokes, a turbine runner with many vanes, or a dynamo armature; N is called the *gyroscopic moment*. The negative sign shows that the lower left and upper right bearings furnish the reactions. In general, since the sum of two principal moments of inertia is always greater than the third—this fact following immediately from the definition Σmr^2—the gyroscopic moment, (1), about a principal axis, fixed or not, is negative for positive n and θ. Hence the direction of the moment follows the rule in Theorem X, (iii). As the result is useful we shall state it formally.

RULE. ***θ turns the spin vector n toward the gyroscopic moment vector N, both N and n being drawn in the same sense.*** (See Fig. 24.) ***N acts on the rotor; the axes must be principal.***

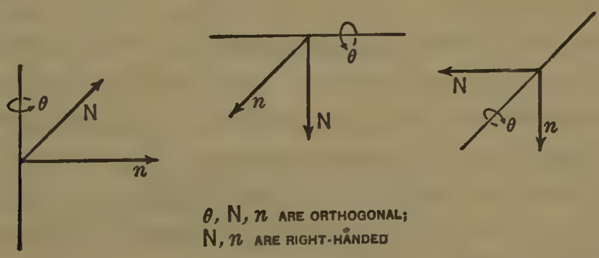


FIG. 24

Of course θ is no more the cause of N than N is of θ; the rule merely gives the relation between them. Huntington[1] has put it into a convenient, easily applied form for the case in which the direction of θ is to be found from that of N, and conversely.

Imagine N as produced by the push of a rough board against the axle (axis of n). For example, in Fig. 24, since N is right-handed, the push against the n vector in the three cases is respectively downward, toward the left, and toward the reader. The direction of θ is then that in which the axle would be rolled by friction. When the direction of θ is known, the rule gives the corresponding direction of N. In both cases N is the moment on —not by—the rotor.

Ex. 77. Show that the wheels of a car making a left turn tend to tip it outward; that a ship with a rotor on longitudinal axis running clockwise when viewed from aft is slewed (turned about vertical) toward the left when the ship pitches bow down; that the apex of a top viewed from above describes a curve in the same sense as the top spins about its axis of figure.

Ex. 78. N varies when the rotor is not a solid of revolution. In the case of an airplane propeller with two blades assume $A = B = kC$ and show that in Fig. 23

$$N = -Cn\theta, \qquad \textit{blades horizontal,}$$
$$N = -(2k-1)Cn\theta, \qquad \text{“} \quad \textit{vertical.}$$[2]

Ex. 79. Show the error in the statement "Since the angular momentum axis (shaft) in Fig. 22 is not changing its direction there is no gyroscopic moment."

The momentum *components* are changing direction.

28. Steady Motion about a Fixed Point. Figure 25 shows a rigid body pivoted at a point and turning around the vertical at constant speed and constant inclination. The problem is to find the relation between θ and φ. To solve it from the standpoint of gyroscopic theory, take axes as shown: X through the centroid and Z horizontal and toward the reader. The axes are fixed to the body and turn with it. Let them be *principal;* this is then a special case, but it will serve as an illustration of the method.

[1] Engineering News, July 21, 1910.

[2] See Messick's U. S. Patent 1491997, 1924, for a method of overcoming the vibrations due to variations of N.

$B\theta \sin\varphi$ about $+Y$ turns at the rate $\theta\cos\varphi$ about X toward $-Z$, the corresponding vector rate of change of angular momentum being

$$-B\theta^2 \sin\varphi\cos\varphi;$$

the sign is that of the axis, $-Z$, *toward which the arrow of the momentum vector moves.*

$A\theta\cos\varphi$ points along $-X$ because $\theta\cos\varphi$ turns from Z to Y, *i.e.*, left-handed. The component $\theta\sin\varphi$, about Y, rotates the momentum arrow toward $+Z$, whence the change of momentum is about $+Z$.

The moment of W about Z being left-handed, we have

$$-Wr\sin\varphi = -B\theta^2\sin\varphi\cos\varphi + A\theta^2\cos\varphi\sin\varphi,$$

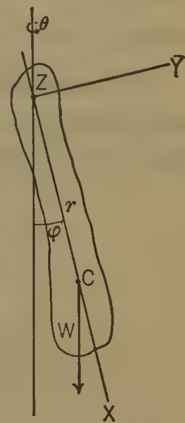


FIG. 25

or

$$Wr = (B - A)\theta^2\cos\varphi,$$

which is like the equation for N, p. 46, because the resultant or total angular momentum is vertical and thus fixed in direction.

Ex. 80. The equation $N = I\alpha$, p. 35, cannot be used in Fig. 25 because W is not parallel to the (horizontal) plane of motion; but the principle from which it was derived is applicable. From the fact that the sum of the moments, about Z, of all the external forces equals the sum of the moments of every element of mass multiplied by its linear acceleration, derive the result in the text.

Note that the acceleration of any point (x, y) is $(x\sin\varphi + y\cos\varphi)\theta^2$, its lever arm being $x\cos\varphi - y\sin\varphi$.

If B_c refers to a centroidal axis parallel to Y, the equation for Wr is, Ex. 64, p. 40,

$$Wr = (B_c - A + mr^2)\theta^2\cos\varphi, \qquad B_c + mr^2 = B,$$

r being taken as always positive; if the centroid is above the point of support, r is still positive but $\varphi > 90°$.

There are two cases to be considered:

(i) $B_c = A$; hence $r\theta^2\cos\varphi = g$.

This is the same result as for the simple conical pendulum. The angle φ increases with r for a given speed $\dot{\theta}$. For $g > r\dot{\theta}^2$ there is no value of φ compatible with *steady* motion.

(ii) If $B_c \mp A$, let $B_c = A \pm m\epsilon^2$; hence $(r^2 \pm \epsilon^2)\dot{\theta}^2 \cos\varphi = rg$.

For steady motion to be possible, $rg \not> (r^2 \pm \epsilon^2)\dot{\theta}^2$; for otherwise $\cos\varphi > 1$.

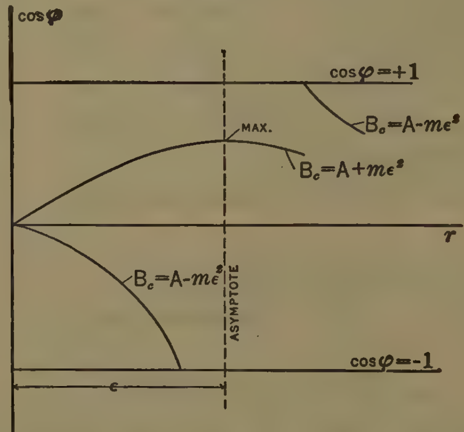


Fig. 26

When $B_c = A + m\epsilon^2$ Fig. 26 shows that φ (not $\cos\varphi$) is least for $r = \epsilon$ and greatest (90°) when the pivot is at the centroid ($r = 0$).

When $B_c = A - m\epsilon^2$ and $r < \epsilon$ the body will be in steady motion at $\varphi > 90°$, $\cos\varphi$ negative, *i.e.* with its centroid *above* the point of support. In this case it makes no difference whether $\varphi > 180°$ (centroid to left of vertical) or $\varphi < 180°$ (centroid to right of vertical), since the distinction between right and left depends on the position of the observer. $\varphi = 90°$ when the pivot is at the centroid. The graph stops at the horizontal unity lines.

Ex. 81. If the body in Fig. 25 is a thin rectangular plate 3 x 5 inches with its plane vertical, find φ when the pivot is (a) on the long axis of symmetry and 0.5, 1.5 inches from the center, (b) on the short axis and 0.5, 1.5 inches from the centroid. Take $g = 32$ and $\dot{\theta} = 32$ rad. per sec.

Ex. 82. Find the equation for φ when the pivot in Fig. 25 is at a distance c from the vertical axis.

With the origin at the pivot and axes oriented as in Fig. 25 we get by means of (4), p. 38,

$$Wr = \frac{W}{g} rc\dot{\theta}^2 \operatorname{ctn} \varphi + (B - A)\dot{\theta}^2 \cos \varphi,$$

φ being positive away from the vertical.

In the railroad between Barmen and Elberfeld, Germany, the cars are suspended from a single rail. The above equation applies to this case if the gyroscopic action of the wheels and motors is neglected: c is the radius of curvature of the track. It will be evident from Fig. 26, when modified to suit this case, that unless c, r, and $(B - A)$ are properly adjusted the cars may incline very much or very little, both circumstances being uncomfortable for passengers.

Ex. 83. Derive the result in Ex. 82 when the origin of the axes is at the centroid.

Ex. 84. A string of length l with a gyro at the end turns about the vertical in steady motion, as in Fig. 27. Find λ, μ.

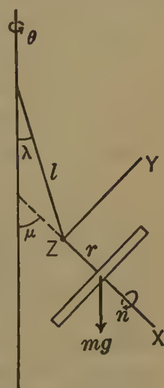


FIG. 27

Let the axes move with the system so that the xy plane remains vertical; the axes are principal for a solid of revolution.

The initial spin n of the rotor remains constant unless retarded by friction which is to be neglected. The motion of the axes cannot change n because the rotor is free to turn on its axle. By the rule on p. 49 the direction of the precession (motion of the axle) will be as indicated.

The equations of moments may be written by means of Theorem X, p. 33, but for variety we shall use its algebraic equivalent, (1), p. 32:

$$\begin{array}{lll}
\omega_1 = n, & \omega_2 = \dot{\theta} \sin \mu, & \omega_3 = 0, \\
\theta_1 = -\dot{\theta} \cos \mu, & \theta_2 = \dot{\theta} \sin \mu, & \theta_3 = 0, \\
h_1 = An, & h_2 = B\dot{\theta} \sin \mu, & h_3 = 0, \\
L_0 = 0, & M_0 = 0, & N_0 = -m\dot{\theta}^2 lr \sin \lambda \cos \mu, \\
L = 0, & M = 0, & N = -mgr \sin \mu.
\end{array}$$

$$\therefore \quad -mgr \sin \mu = -mlr\dot{\theta}^2 \sin \lambda \cos \mu - An\dot{\theta} \sin \mu - B\dot{\theta}^2 \sin \mu \cos \mu.$$

The force equations are

$$T \cos \lambda = mg, \qquad T \sin \lambda = m(l \sin \lambda + r \sin \mu)\dot{\theta}^2.$$

No other equations are possible and only three unknowns may be found; therefore $\dot{\theta}$ must be given if λ, μ, T are unknown.

Ex. 85. For what value of n, *and why*, will the result in Ex. 84 agree with that in Ex. 82?

Ex. 86. (a) Has the rotor angular velocity when $n = 0$?

Ex. 87. When will θ be real in Ex. 84? Is it possible for μ to be negative? When will $\mu = 90°$?

Ex. 88. Solve Ex. 84 when the origin is at the centroid.

29. Rolling. If a disk is to roll along a curved path it must lean inward to be in steady motion or in what might be called quasi-equilibrium. It is for this reason that railroad and running tracks are banked on the curves, but the banking must be greater when the gyroscopic action of the wheels is included.

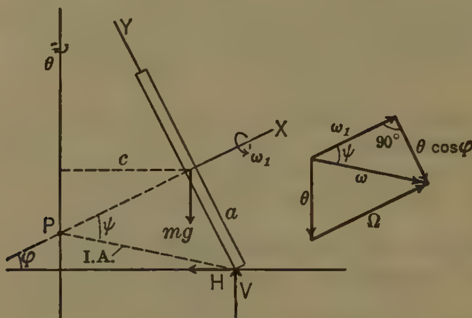


FIG. 28

In Fig. 28, which shows a circular disk, the ground reactions are H, V; the center of the disk describes a circle of radius c at θ radians per second. There is pure rolling, *i.e.*, no slipping.

Let the XY plane remain vertical. Since the axes are not fixed to the body, only the origin being so fixed, Euler's equations, p. 38, cannot be used.

Call the *absolute* spin of the body ω_1; this is what is commonly meant in saying that the disk is turning at a certain number of radians per second. The disk has also the velocity $\omega_2 = -\theta \cos \varphi$, but $\omega_3 = 0$.

The rolling body (not the axes) has two instantaneous centers, two points whose velocities are zero: one at the ground and another, P, at the intersection of the axle and the vertical. The instantaneous or resultant velocity axis $I.A.$ passes through them. The ω vector, along $I.A.$, has components ω_1 and ω_2 and

also components θ and the relative velocity Ω: compare Figs. 28 and 14.

In (1), p. 32, $\theta_1 = -\theta \sin\varphi$, $\theta_2 = -\theta\cos\varphi$. Hence

$$Va\sin\varphi - Ha\cos\varphi = A\omega_1\theta\cos\varphi + B\theta^2\sin\varphi\cos\varphi,$$

from which H, V are to be eliminated by means of

$$V = mg, \qquad H = mc\theta^2.$$

Since the right member of the moment equation is positive, we have $H < V\tan\varphi$; hence the resultant of H and V passes to the right of the centroid and the disk leans more, *for the same value of c*, than it would if the gyroscopic action (the A and B terms) were neglected.

When ω_1 is eliminated from the moment equation by means of the relation $c\theta = a\omega_1$, we get

$$mga^2\sin\varphi = (Ac + Ba\sin\varphi + mca^2)\theta^2\cos\varphi.$$

It will be shown in Chapter VII that this motion is stable.

Ex. 89. If the axle in Fig. 28 is pivoted to a support at P, the ground reaction H may be replaced by a left horizontal force Q (not shown) exerted by the pivot on the axle; there will also be a vertical force R at P acting, say, downward, which however does not make V unnecessary. Show that the equation of moments is

$$Va\sin\varphi + Vc - mgc - mc^2\theta^2\tan\varphi = (A\omega_1 + B\theta\sin\varphi)\theta\cos\varphi.$$

What part of V is due to gyroscopic action?

If V_0 is V when gyroscopic action is omitted, *i.e.* when the moments of inertia are made zero,

$$V_0(c + a\sin\varphi) = mgc + mc^2\theta^2\tan\varphi.$$

Hence

$$V - V_0 \text{ is due to gyroscopic action.}$$

Ex. 90. The device in Fig. 29 is used for crushing rock in cement mills. Show that the least speed must be greater than

$$\theta^2 = \frac{mgra}{Ar\sin\varphi + Ba\cos\varphi}.$$

Proceed as in the solution of Fig. 28.[1]

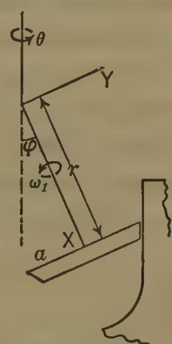


FIG. 29

[1] For an investigation of the most effective type of crusher see Grammel, *Zeitschrift des Verein. deutscher Ingenieure*, 1917, p. 572; Grammel, *Der Kreisel*, § 14.

30. Banking of Tracks. Tracks and roads are banked on curves in order to avoid flange pressure which tends to spread the rails, or to prevent skidding when there are no rails.

In Fig. 30 a car, running along $+Z$ toward the reader, is

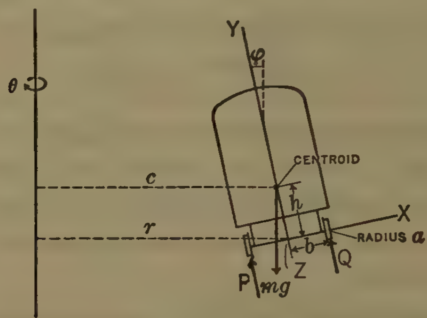


Fig. 30

rounding a curve of radius r, the centroid describing an arc of radius c at velocity θ. When there is no flange reaction the three unknowns P, Q, φ are found from three equations, two of which are

$$P + Q - mg\cos\varphi = mc\theta^2\sin\varphi, \tag{1}$$

$$mg\sin\varphi = mc\theta^2\cos\varphi. \tag{2}$$

Equation (2) gives φ correctly only when neither P nor Q is negative; h should be made less when one of the reactions is a pull instead of a push.

To write the moment equation about Z, let I be the moment of inertia of the wheels about X, A that of the car without wheels about X, B that of the car with wheels about Y. It is assumed that all the wheels in action are replaced by a single pair. Then

$$\begin{aligned} Qb - Pb + mgh\sin\varphi = {} & mr\theta^2 h\cos\varphi \\ & + (I\omega_1 - A\theta\sin\varphi)\theta\cos\varphi + (B\theta\cos\varphi)\theta\sin\varphi; \end{aligned} \tag{3}$$

the parentheses are the angular momenta. The term $mr\theta^2 h\cos\varphi$ is N_0, p. 33; $\omega_1 a = \theta c$.

The moment of the tractive force has been omitted. If the tractive force T at the drawbar is inclined horizontally at ψ to a tangent to the path and acts at a height k, there must be added to the left member

$$(T \sin \psi \cos \varphi)k.$$

As φ is generally not more than 20° it is sufficiently precise to call $\cos \varphi$ unity in order to estimate the effect of gyroscopic action. The discussion of the equation is left as an exercise for the student.

An important gyroscopic phenomenon still remains to be studied in this connection. As the outer wheel runs from the level to the elevated part of the rail, the car and wheels get an angular velocity $\dot{\varphi}$ about $+Z$. In Fig. 31 the outer wheel is shown as rising in time dt through a distance that is approximately $l\ d\varphi$ since φ is not large; it is climbing a grade of angle ψ at velocity v. Therefore

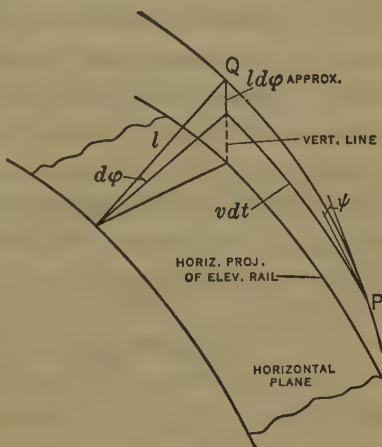


FIG. 31

$$l\, d\varphi = v\, dt \tan \psi, \qquad \text{or} \qquad l\dot{\varphi} = v \tan \psi.$$

Now $\dot{\varphi}$ turns the angular momentum $I\omega_1 - A\theta \sin \varphi$ toward $+Y$; consequently the rails must exert a moment

$$(I\omega_1 - A\theta \sin \varphi)\frac{v}{l} \tan \psi,$$

about Y, on the wheel flanges. Thus the rear end of the car tends to slew outward and the front end inward, the opposite taking place as the track again becomes level. This shows the importance of long and gradual banking.

31. The Gyroscopic Horizon. A sextant is used to measure the angular altitude of the sun, or any other star, above the horizon. When the actual horizon is not visible the angle between the star and its image in a horizontal reflecting surface is measured; on land this may be a pool of mercury but the motion of a ship makes the use of such an artificial horizon impossible. A mirror mounted on a pendulum having a pivot or point support can also be used on land but evidently not aboard a moving vessel.

In 1752 Serson experimented with a top whose upper surface was a polished metal disk; he seemed to be under the impression that a spinning body will maintain the direction of its axis. But unless the point of support is at the exact centroid of the rotating body the axis of spin will precess, *i.e.*, will describe a cone about the vertical through the point of support. Admiral Fleuriais of the French Navy was the first to embody Serson's idea in a practical instrument.[1]

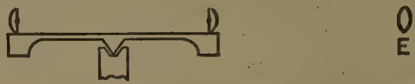


FIG. 32

A rotor, Fig. 32, driven by compressed air, or electrically in later forms, is pivoted on a support attached to the frame of a sextant; *E* is the eye-piece of the telescope. Two plano-convex lenses are mounted on the rotor, plane faces toward each other; a horizontal black center line is engraved on each plane face. The lenses are so spaced that both lines—the farther line being seen through the nearer lens—are in focus for the eye.

The mass of the rotor is 175 gm., the centroid is 1 mm. below the point of support, the speed is at least 300 rad. per sec., the velocity of precession is about 0.05 rad. per sec.

During the spin of the rotor, the entire circular path of the black marks remains visible on account of the persistence of vision. The projection of this circle on a plane normal to the

[1] *Revue maritime et coloniale*, 1886, vol. 91, p. 452, and 1910, vol. 184, p. 5.

line of sight is a line or a narrow ellipse, depending on the tilt of the spin axis. For a right-handed (west to east) rotor spin, the precession of the axis is left-handed about the vertical since the centroid is below the pivot: see rule of signs, p. 49.

If the telescope is horizontal and the spin axis inclined directly away from observer, he sees the central part of the ellipse (iii), Fig. 33, the upper arrow showing the direction of rotation of the nearer mark. Figures 33 (i)–(v) show the projected circle at 45° intervals of left-handed precession. The cycle illustrated takes about a minute. The major axis of the ellipse (iii) is used as the horizon in the sextant observations. Skill is required to detect the major axis, to level the telescope, and to adjust the sextant mirror promptly.

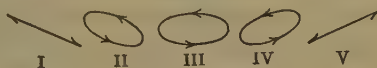


FIG. 33

In the following discussion the moments L_0, M_0, N_0, p. 33, which are produced by the motion of the ship, will be neglected because the centroid is very near the pivot point. With the origin of axes at the pivot, Y upward along the axle, X toward the right, and Z toward the reader, as in Fig. 34, the equation of moments is

$$-mgh\sin\psi = -Bn\theta\sin\psi - A\theta^2\sin\psi\cos\psi,$$

where ψ is the angle at which Y is inclined to the left of the vertical in case (ii).

Since θ is small, its value being 0.05, we have

$$mgh = Bn\theta,$$

h being 1 mm.; ψ is thus almost independent of θ and n.

Ex. 91. Find B, using the data on p. 58.

The rotation of the earth has been neglected in this discussion. It would have no effect on a gyro supported at its centroid, but when the centroid is not at the point of support the centroid is constrained by gravitational pull to participate in the rotation,

In Fig. 34 Y is the axle of the gyro, ψ being its deviation from the vertical. Let us now find the effect of the earth's rotation at the instant the XY plane is in the meridian; the general case will be taken up in Chapter VIII without the restriction that ψ is constant.

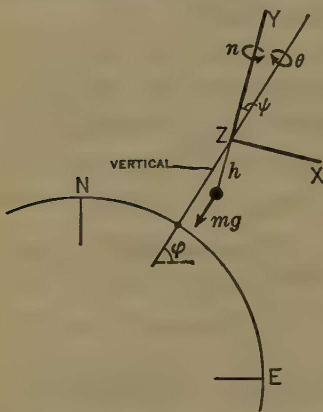


FIG. 34

Here n is the absolute spin of the gyro—not, as usually measured, the spin relative to the earth; for large n the distinction has no numerical importance. θ is the *relative* precession of the axle.

It is of great importance to understand the effect of ω, the angular velocity of the earth. Its Y component, $\omega \sin(\varphi + \psi)$, has no effect on the spin of the gyro, but it rotates the X angular momentum. The X component, $\omega \cos(\varphi + \psi)$, is an angular velocity of the gyro and gives rise to X angular momentum.

The moment equation is found thus:

Bn turns at $(\theta \sin \psi + \omega \cos(\varphi + \psi))$ about $+X$ toward $-Z$;

$A(\theta \sin \psi + \omega \cos(\varphi + \psi))$, which is right-handed about $-X$, turns left-handed at $(\theta \cos \psi - \omega \sin(\varphi + \psi))$ about $+Y$ toward $-Z$;

$mgh \sin \psi$ is negative.

Hence

$$mgh \sin \psi = Bn\theta \sin \psi + Bn\omega \cos(\varphi + \psi)$$
$$+ A\theta^2 \sin \psi \cos \psi + A\theta\omega \cos \psi \cos(\varphi + \psi)$$
$$- A\theta\omega \sin \psi \sin(\varphi + \psi) - A\omega^2 \sin(\varphi + \psi) \cos(\varphi + \psi).$$

This equation is approximate in spite of its length: ψ is assumed constant and the axle is in the plane of the meridian. Since θ, ω, ψ are small, we have

$$mgh \sin \psi = Bn\theta \sin \psi + Bn\omega \cos \varphi.$$

To find the effect of ω, put $\theta = 0$ and $\psi = \psi_0$; then

$$\sin \psi_0 = \frac{Bn\omega \cos \varphi}{mgh},$$

where ψ_0 is the deviation from the true vertical produced by the earth's rotation. On p. 59, $Bn\theta$ is approximately mgh. Then

$$\sin \psi_0 = \frac{\omega \cos \varphi}{\theta}.$$

Ex. 92. Show that $\psi_0 = 0.08°$ at the equator.

The latitude as determined by means of the gyroscopic horizon is therefore $\psi_0°$ *more* than the true latitude if the gyro spins right-handed as the earth does, and if the centroid is below the point of support.

CHAPTER V

THE FREE GYRO

32. Free Gyro. Any rotating body will be called a *gyro*;[1] it is said to be *free* when no external moments act on it to change its angular momentum and its kinetic energy of rotation: all external forces are concurrent at its centroid. Actually there is no such thing as a free gyro, but in certain cases, such as the earth, a projectile, a base-ball, etc., the disturbing moments are so small as to alter only slightly the character of the motion.

For a free gyro, the angular momentum h and the kinetic energy *of rotation* are constant and are connected by the relation, Ex. 65, p. 40,

$$2E = h\omega \cos \varphi.$$

Therefore $\omega \cos \varphi$, the angular velocity about the h axis, is also constant. But it does not follow that ω is constant; on the contrary, it is seen from Ex. 58, p. 38, that φ in Fig. 20 cannot be constant when the principal moments of inertia A, B, C are different.

Throughout this chapter A, B, C represent principal moments of inertia *arranged in decreasing order of magnitude.*

Ex. 93. Prove that if there is a resultant moment about an axis perpendicular to the plane of h and h_i (see p. 39, for h_i), $\omega \cos \varphi$ and E are both constant; see Ex. 66, p. 40.

33. Invariable Plane. It is evident from Fig. 35 that the ω vector terminates in a plane that is normal to the fixed direction of h at a distance $2E/h$ from the centroid of the gyro. This plane is fixed in *direction* even if the centroid has acceleration or velocity; it is called the *invariable plane.*

[1] This convenient term is used in patents to denote the rotor of any gyroscopic device and occurs also in such combinations as *gyro-compass*, *gyro-stabilizer*. The word *gyroscope* was coined by Foucault in 1852 and means etymologically *to see rotation.* English writers, following Kelvin, call it a *gyrostat* because the axis tends to remain stationary.

From

$$\omega \cos \varphi = constant,$$

$$\tan \varphi = \frac{d\omega}{\omega d\varphi},$$

which suggests that $2E/h$ is perpendicular to the tangent line (in the h, ω plane) to some curve of which ω is the radius vector. Now the components of ω are connected by (see Ex. 65 and § 23)

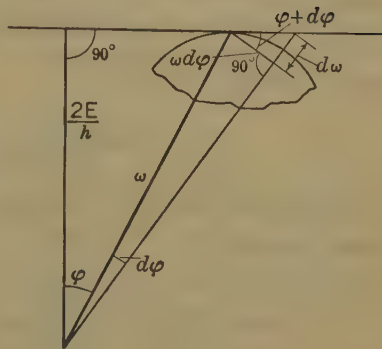


FIG. 35

(1) $$A\omega_1^2 + B\omega_2^2 + C\omega_3^2 = 2E = \text{const.},$$

(2) $$A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 = h^2 = \text{const.};$$

therefore if ω_1, ω_2, ω_3 are used as Cartesian coordinates, ω is the radius vector to each ellipsoid (1) and (2). The surface (1) is called the *energy ellipsoid*, and (2) the *momentum ellipsoid*. Let us see if one of them is tangent to the invariable plane. In order that a plane with direction cosines λ, μ and ν, that is,

$$\lambda x + \mu y + \nu z = d,$$

may be tangent to an ellipsoid

$$ax^2 + by^2 + cz^2 = 1,$$

each of the slopes $\partial y/\partial x$, $\partial z/\partial x$ must have identical values for both the plane and the ellipsoid at the point of tangency (ξ, η, ζ); this gives

$$a\mu\xi = b\lambda\eta, \qquad a\nu\xi = c\lambda\zeta.$$

When these are used to eliminate μ and ν from the equation of the plane, we find

$$a\lambda\xi x + b\lambda\eta y + c\lambda\zeta z = ad\xi,$$

which gives, at the point (ξ, η, ζ),

$$ad\xi = \lambda,$$

since (ξ, η, ζ) lies on the ellipsoid. Hence the preceding equation becomes

$$a\xi x + b\eta y + c\zeta z = 1.$$

If α, β, γ are the direction cosines of the perpendicular p to this plane,

$$\alpha = pa\xi, \qquad \beta = pb\eta, \qquad \gamma = pc\zeta.$$

Since $\alpha^2 + \beta^2 + \gamma^2 = 1$,

$$p^2 = 1 \Big/ \frac{1}{a^2\xi^2 + b^2\eta^2 + c^2\zeta^2}.$$

But we saw that $2E/h$ is a perpendicular to a possible tangent plane, where

$$\left(\frac{2E}{h}\right)^2 = 1 \Big/ \frac{1}{\left(\dfrac{A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2}{(2E)^2}\right)}.$$

Now if p is the perpendicular $2E/h$, and ξ, η, ζ stand for ω_1, ω_2, ω_3, the coefficient a in p^2 corresponds to $A/2E$, b to $B/2E$, c to $C/2E$. The ellipsoid which is tangent to the invariable plane is thus the *energy ellipsoid* (1).

The energy ellipsoid, being constructed on the principal axes of the body, moves with it. The velocity of the ellipsoid, and of the body, about the h axis is the projection of ω on h; consequently the ellipsoid rolls without slipping on the invariable plane. For if there were slipping this would cause an additional angular velocity about h. Or consider the extreme case of sliding without any rolling: the resulting angular velocity of the ellipsoid would certainly not be about the radius vector to the point of contact.

The path traced on the tangent plane by the point of contact is called the *herpolhode* and is analogous to the space centrode of plane kinematics; the locus of the point of contact on the ellipsoid itself is the *polhode* and corresponds to the body centrode. These terms are due to Poinsot (1834), who originated the ideas in this chapter.

34. The Polhodes. A polhode is the locus of the end of the ω vector on the surface of (1), p. 63. Since the ω components must also satisfy (2), the polhode is the curve of intersection of the two ellipsoids; ω_1, ω_2, ω_3 are used as Cartesian coordinates measured along the principal axes. The properties of the polhode may be studied from its projections on the principal planes. The elimination of ω_1, ω_2, ω_3, one at a time, from (1) and (2) gives

$$B(B - A)\omega_2^2 + C(C - A)\omega_3^2 = h^2 - 2EA, \tag{3}$$

$$C(C - B)\omega_3^2 + A(A - B)\omega_1^2 = h^2 - 2EB, \tag{4}$$

$$A(A - C)\omega_1^2 + B(B - C)\omega_2^2 = h^2 - 2EC. \tag{5}$$

The perpendicular $2E/h$, Fig. 35, obviously lies between the greatest and least semi-axes of the energy ellipsoid; hence if

$$A > B > C,$$

it follows that

$$\sqrt{\frac{2E}{A}} < \frac{2E}{h} < \sqrt{\frac{2E}{C}}, \qquad \text{or} \qquad 2EA > h^2 > 2EC.$$

When a principal axis coincides with the invariable perpendicular the proper inequality sign is replaced by an equality sign.

On the assumption that $A > B > C$ the signs of the terms in (3)–(5) show that the polhode projections are ellipses on the BC and AB (or $\omega_2\omega_3$ and $\omega_1\omega_2$) planes and hyperbolas on the AC (or $\omega_1\omega_3$) plane. When $h^2 = 2EB$ the polhode is a pair of ellipses whose projections on the $\omega_1\omega_3$ planes are the asymptotes of the hyperbolas in (4). They are called the *separating* polhodes.

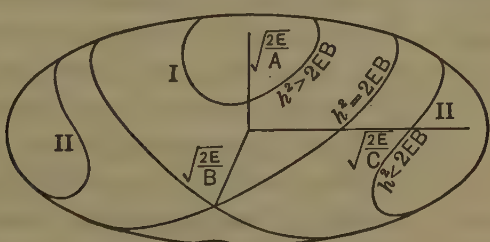


Fig. 36

Figure 36 shows the energy ellipsoid with polhodes for three values of h; the momentum ellipsoids are to be imagined as passing through the polhodes.

To distinguish between the polhodes i, ii, notice that when $h^2 > 2EB$ the vertices of the hyperbolas in (4) lie on the ω_3 or C axis; therefore $h^2 < 2EB$ for a polhode of type ii.

Ex. 94. Show that the planes of the separating polhodes are at 90° when $A(A - B) = C(B - C)$.

Ex. 95. A body for which $A = 5$, $B = 4$, $C = 3$ is given a spin of 12 rad. per sec. about a line at 45° to each of the principal axes Find E, h, φ and the polhode.

Ex. 96. Draw the polhodes when $B = C$.

35. The Angular Momentum Loci. Equations (1) and (2) on p. 63 may be written

$$\frac{h_1^2}{A} + \frac{h_2^2}{B} + \frac{h_3^2}{C} = 2E, \tag{6}$$

$$h_1^2 + h_2^2 + h_3^2 = h^2. \tag{7}$$

If h_1, h_2, h_3 are used as coordinates along the principal axes, the end of the angular momentum vector h terminates in the intersection of ellipsoid (6) and sphere (7). These intersections, Fig. 37, which might be called angular momentum loci, look like polhodes but are not. They show how the axis of resultant angular momentum wanders about in the body.

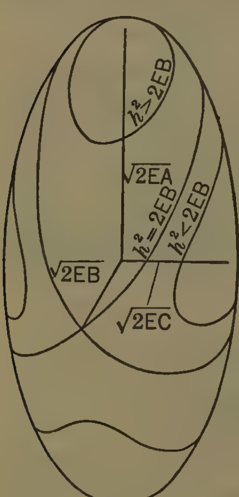


Fig. 37

36. Stability of Rotation. A body in equilibrium has statical stability when it will recover its position after a slight disturbance has been given to it. The disturbance must be small because a large departure from the critical position may take the body to a field of radically different force properties. Similarly a body rotates stably about an axis, *i.e.*, has *kinetic*

or *dynamic* stability, when a small disturbance will not cause the ω axis to depart much from its initial position. Vibrating or oscillating systems may also have kinetic stability; see Chapter VIII.

When a free gyro spins about its principal A axis

$$\omega_2 = \omega_3 = 0, \qquad h^2 = 2EA, \qquad \frac{2E}{h} = \sqrt{\frac{2E}{A}}.$$

Thus the polhode, Fig. 36, and the momentum locus, Fig. 37, degenerate to a point. Likewise when $h^2 = 2EC$. Hence when a free gyro spins about an axis of greatest or least moment of inertia the position of the ω– and the h–axis remains fixed in the body. It remains fixed in space also, because since the energy ellipsoid (1), p. 63, is tangent to the invariable plane at the end of its least or its greatest axis, it is impossible for the ellipsoid to roll away from its point of tangency on the plane; *i.e.*, the herpolhode is a point and the axis of spin stays fixed in direction. For this reason the A and C axes are called *permanent.*

When the spin is about the intermediate or B axis of inertia, $h^2 = 2EB$, and the ω axis moves completely around the energy ellipsoid along one of the separating polhodes. This means that the body, for example a rotating projectile, will finally spin about an axis at 90° to its initial axis. Even when $h^2 \neq 2EB$, the axis of spin may turn through 90° from its initial position provided h^2 is nearly $2EB$, the polhode in this case being a long curve—a long projected ellipse.

When two of the moments of inertia, *e.g.* B and C, are nearly equal, the polhode around the third (the A axis) is nearly circular, so that the precession is nearly constant. On the other hand, if the polhode about A is much elongated,—B much different from C—the ω axis may wander so far from the A axis as to enter a region of disturbances not present near A.

The polhodes in Fig. 36, due to Poinsot, show the facts stated above, but do not explain why the ω axis changes its position in

space. However, the reason for its behavior follows from the phenomena discussed in §§ 24, 25: it is either the products of inertia or the last terms in Euler's equations, Ex. 58, that turn the ω axis.

37. Disturbed Stable Rotation. Let a body that has been given a spin n about its axle, say the A axis, be slightly disturbed by having *small* velocities ω_2 and ω_3 impressed on it by moments applied for a short time and then removed. The product $\omega_2\omega_3$ is negligible in (4), p. 38. Then

$$A\dot{\omega}_1 = 0, \quad \text{or} \quad \omega_1 = \textit{constant} = n,$$

(1) $$B\dot{\omega}_2 - (C - A)n\omega_3 = 0,$$

(2) $$C\dot{\omega}_3 - (A - B)n\omega_2 = 0.$$

Differentiate these equations and eliminate ω_2 and ω_3 one at a time:

(3) $$\ddot{\omega}_2 = -\lambda^2\omega_2,$$

(4) $$\ddot{\omega}_3 = -\lambda^2\omega_3,$$

where

(5) $$\lambda^2 = \frac{(A - C)(A - B)}{BC}n^2,$$

λ^2 being positive when B is the intermediate moment of inertia. Since the equations are of the simple harmonic type, ω_2 and ω_3 fluctuate according to sine or cosine laws, and the instantaneous axis oscillates in the neighborhood of its initial position. The motion is therefore stable. In other words, a free gyro spinning rapidly about its axis of greatest or least moment of inertia gets only a slight wobble or a slight precession when a small velocity is imparted about some other axis. But if the body is not spinning at all, or so slowly that any disturbance is relatively large, any impressed velocity, *e.g.*, ω_2, would turn it completely around the ω_2 axis. This explains why guns are rifled and why a spinning playing card can be thrown far, whereas a non-spinning card cannot be thrown at all.

If A were the intermediate moment of inertia, $+\lambda^2$ would be replaced by $-\lambda^2$, the right-hand members of the differential equations would be positive, and the motion would not be harmonic. In this case the equations (3) and (4) will be satisfied by $\omega = e^{\pm\lambda t}$. Since each of these differential equations is of the second order, the complete solution of each of them requires two constants, and we may write

$$\omega_2 = c_1 e^{\lambda t} + c_2 e^{-\lambda t},$$
$$\omega_3 = c_3 e^{\lambda t} + c_4 e^{-\lambda t}.$$

Since λ is real, the velocities increase with the time. But they cannot do so indefinitely, because the differential equations are only approximately true and cease to apply when ω_2 and ω_3 become comparable in value with ω_1. Nevertheless they show the motion to be unstable even if they do not specify it precisely.

Ex. 97. Discuss the motion for $C > B > A$ with A nearly equal to B.

38. Unstable Rotation. We have just seen that the rotation is unstable when the h vector or the ω vector is at the intermediate axis B. The case of rotation about the B axis (when $h^2 = 2EB$) is interesting because it is not easy to foresee why the principal axes should have different properties in regard to stability. Furthermore, it happens to be one of the few instances of the general problem that is integrable without the aid of elliptic functions.

When there are no applied moments, the second of Euler's equations, p. 38, is

$$\dot{\omega}_2 = \frac{C - A}{B}\,\omega_1\omega_2. \tag{1}$$

To express ω_1 and ω_3 in terms of ω_2, use (3), (5), p. 65. This gives

$$\omega_1^2 = \frac{B - C}{A - C}\cdot\frac{h^2 - B^2\omega_2^2}{AB},$$

$$\omega_3^2 = \frac{A - B}{A - C}\cdot\frac{h^2 - B^2\omega_2^2}{BC},$$

whence

$$a\frac{h}{B}\,dt = \frac{d\left(\dfrac{B\omega_2}{h}\right)}{1 - \left(\dfrac{B\omega_2}{h}\right)^2},$$

in which

$$a = \pm\sqrt{\frac{(A-B)(B-C)}{AC}}.$$

The integral is [1]

$$\omega_2 = \pm\frac{h}{B}\tanh\frac{ah}{B}(t + t_0),$$

where t_0, the constant of integration, depends on the initial conditions. The sign of ω_2, which is + or − on account of the double sign of a, is to be chosen to fit $\dot{\omega}_2$ in (1); *i.e.*, since $C - A$ is negative ($A > C$), the sign of $\dot{\omega}_2$ will be negative if ω_1 and ω_2 have like signs.

Since $1 - \tanh^2 x = \operatorname{sech}^2 x$, we get

$$\omega_1^2 = \pm\frac{B-C}{AB(A-C)h^2}\operatorname{sech}\frac{ah}{B}(t + t_0),$$

$$\omega_3^2 = \pm\frac{A-B}{BC(A-C)h^2}\operatorname{sech}\frac{ah}{B}(t + t_0).$$

As t approaches infinity, the hyperbolic tangent approaches unity and the secant zero; thus the ultimate value of ω_2 is $\pm h/B$, that of ω_1 and ω_3 being zero. The energy ellipsoid then touches the invariable plane at the end of the B axis. But ω_1 and ω_3 never really vanish, and the intermediate axis, ω_2, is never the instantaneous axis and never becomes perpendicular to the invariable plane. The herpolhode winds about and continually approaches the invariable line; but its length cannot be infinite, being obviously equal to the length of the separating polhode. Now a spiral of the type $r = e^{-\theta}$ winds around and always gets nearer the origin and has finite length: its length from $\theta = 0$

[1] Peirce, *Short Table of Integrals*, No. 687.

to ∞ is $\sqrt{2}$. The herpolhode has in fact been shown by Darboux to be a logarithmic spiral; the proof is too long to be given here.

39. Air Resistance. The moment of the air drag on a gyro varies as some power of the angular velocity, the exponent lying between *one* and *two*. The resistance is of a two-fold character; it depends on the skin friction between the surface of the body and the air in contact with it, and on the viscous resistance between contiguous layers of air whirled around at different speeds. If for the sake of mathematical simplicity the exponent is taken as unity, we shall get some idea of the qualitative effect of air resistance although the results may be quantitatively wide of the mark.[1]

For resistance varying as the first power, the moments about axes fixed in the body are

$$L = -k_1\omega_1, \qquad M = -k\omega_2, \qquad N = -k\omega_3,$$

where the last two coefficients are the same if, as we shall assume, the gyro is a solid of revolution. When $B = C$, (4), p. 38, becomes

$$(1) \qquad -k_1\omega_1 = A\dot{\omega}_1,$$

$$(2) \qquad -k\omega_2 = C\dot{\omega}_2 - (C - A)\omega_3\omega_1,$$

$$(3) \qquad -k\omega_3 = C\dot{\omega}_3 - (A - C)\omega_1\omega_2.$$

For an initial spin $\omega_1 = n$, (1) gives

$$\omega_1 = ne^{-(k_1t/A)}.$$

Multiplying (2) by ω_2, (3) by ω_3, and adding, we get

$$\frac{\omega_2 d\omega_2 + \omega_3 d\omega_3}{\omega_2^2 + \omega_3^2} = -\frac{k}{C}dt,$$

where $\omega_2^2 + \omega_3^2 = \omega^2 - \omega_1^2$.

[1] A comprehensive account of gyroscopic action and the air effect (resistance, cushioning, magnus effect) in the case of projectiles is given by Crantz, *Lehrbuch der Ballistik*, vol. I, 5 ed., 1925; see also Wainwright, *Damping effects in exterior ballistics*, Philosophical Magazine, vol. 3, Supplement, Apr., 1927.

Since we are not supposing the initial values of ω_2 and ω_3 to be zero, the initial ω is ω_0, not n; hence

$$\omega^2 - \omega_1^2 = (\omega_0^2 - n^2)e^{-(kt/C)},$$

which shows that ω approaches ω_1 with increasing time. If φ is the angle between ω and ω_1, we have

$$\tan\varphi = \frac{\sqrt{\omega^2 - \omega_1^2}}{\omega_1} = \frac{\sqrt{\omega_0^2 - n^2}}{n} e^{-\left(\frac{k}{C} - \frac{k_1}{A}\right)t}.$$

For a disk-shaped body like a quoit, or an oblate spheroid like the earth,

$$A > C, \qquad k > k_1;$$

hence $k/C - k_1/A$ is positive and φ gradually decreases; *i.e.*, air resistance tends to extinguish the disturbances ω_2 and ω_3, and to make the instantaneous axis ω coincide with the geometric axis ω_1.

For an elongated body like a projectile, $A < C$; and if $k_1/A > k/C$, the exponent of e in $\tan\varphi$ is positive and φ increases with time. Air resistance therefore increases any initial wobble of the projectile, but as ω_1 is generally large, considerable time—more than the time of flight—may be required to make an appreciable change in the position of the ω axis. This shows how the rifling of guns counteracts the tendency to instability set up by friction.

40. The Earth as a Gyro. The attractions of the sun and moon do not pass through the center of the earth. Consider the excess matter lying outside of a sphere constructed on the shorter, polar, diameter: the attraction of the sun and also of the moon on the near part is stronger than that on the far part. A simple diagram will make it clear that, as the earth's axis is inclined at about 66°.5 to the plane of its orbit, this differential attraction results in a moment (greatest at the solstices, zero at the equinoxes) producing left-handed (east to west) precession of the axis about a normal to the ecliptic (orbital plane). (See the Rule of Signs, p. 49.) The solar precession is about 16

seconds of arc a year and the lunar about 34, their sum giving a precession period of about 26,000 years.

The earth makes one turn relative to the fixed stars in one sidereal (star) day of 86,164 seconds, the mean solar (sun) day being 86,400 seconds, *i.e.*, nearly 4 minutes longer. Since there are thus about 366 (not 365) sidereal days in a year, the earth's spin is $366 \times 2\pi$ radians a year. The resultant of this right-handed spin and the *left-handed* precession of $2\pi/26{,}000$ radians a year, the angle between them being 23°.5, is inclined to the earth's axis and toward the ecliptic at about 0.008 second of arc. It is evident from § 39 that although this angle is now nearly zero it might have been large, ages ago, having been reduced to its present value by air and ocean friction.

When the action of small disturbing moments on a gyro ceases, the subsequent motion is defined by (1)–(5), p. 68; this is called the *free* motion because it persists *after* the disturbance that caused it has vanished. In the case of the earth free motion must be attributed chiefly to the action of the sun and moon. With $B = C$ for the earth, (5), p. 68, is

$$\lambda = \frac{A - C}{A} n,$$

where

$$\frac{A - C}{A} = 0.0032,$$

$$n = 0.000073 \textit{ rad. per sec.}$$

Since (3), (4), p. 68, define simple harmonic motions, the period T is given by

$$T\lambda = 2\pi.$$

Hence

$$T = 10 \textit{ months, approximately.}$$

The solution of (3) is easily seen to be

$$\omega_2 = \Omega \cos (\lambda t + c);$$

the two constants Ω, c required by the second order differential equation are to be found from initial or other known conditions.

Substitution in (1), p. 68, gives

$$\omega_3 = \Omega \sin (\lambda t + c).$$

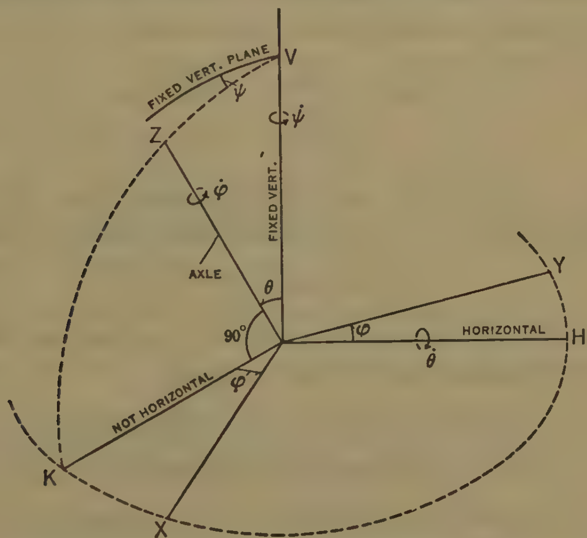
Ω, the resultant of ω_2 and ω_3, lies in the equatorial plane and turns relatively to the earth once in about 10 months; this is called the Eulerian period. The resultant of n, Ω is the real spin axis of the earth. This axis describes a cone of semi-angle arc-tan Ω/n about the axis of figure. Since latitudes are determined from stellar observations made with respect to the spin or instantaneous axis, the wandering of this axis gives rise to variations of latitude. Observations have shown these variations, less than $0.5''$, to have a 14-month period known as the Chandlerian period.[1]

[1] For detailed accounts of the very complex motions of the earth see Routh, *Advanced Rigid Dynamics*, Chap. XI, and Klein and Sommerfeld, *Theorie des Kreisels*, III, Chap. VIII.

CHAPTER VI

THE TOP

41. Moving Axle. The axle of a gyro is a line fixed in the body, about which the gyro gets its initial spin or at any rate its largest spin component; for a solid of revolution the axle[1] is the axis of figure. The chief problem in the theory of the gyroscope is to find the motion of and the forces exerted on or by the axle. For the purposes of this chapter it is accurate enough to regard the earth as non-rotating and to refer all motion to it. In dealing later, Chapter VIII, with the influence of the earth's rotation, the so-called fixed stars will be used as a frame of reference. With this understanding, the orientation of the axle with respect to fixed axes is specified as in Fig. 38.



X, Y, Z ARE PRINCIPAL AXES FIXED IN THE BODY.
H REMAINS HORIZONTAL AND NORMAL TO THE MOVING VERTICAL PLANE V Z K; K, X, Y ARE NOT HORIZONTAL; Z IS NORMAL TO THE PLANE K X H Y

FIG. 38

[1] This convenient term seems to be due to Crabtree, *Spinning Tops and Gyroscopic Motion*, § 32.

Here X, Y, Z are principal axes fixed in the body. The angle θ is the *nutation* angle; its variation causes the nodding (nutation) of the axle in the vertical plane VZK. This plane, inclined at ψ to an arbitrary fixed vertical plane, turns with angular velocity $\dot{\psi}$, called the *precession*. The angle φ orients the body with respect to the moving axes KHZ; $\dot{\varphi}$ is the velocity relative to KHZ.

Since $\dot{\theta}$ and $\dot{\psi}$ are the velocities of the axle and the axes KHZ which move with it, their projections on K, H, Z are

(1) $$\theta_1 = -\dot{\psi}\sin\theta, \qquad \theta_2 = \dot{\theta}, \qquad \theta_3 = \dot{\psi}\cos\theta.$$

The use of θ in two senses need cause no confusion, because the distinction between angle and angular velocity is always obvious.

The angular velocity of the body—not the axle—is the resultant of θ_1, θ_2, θ_3 and $\dot{\varphi}$, see § 8; call its components

$$\omega_1,\ \omega_2,\ \omega_3 \quad \textit{along} \quad K,\ H,\ Z.$$

Then

(2) $$\omega_1 = \theta_1, \qquad \omega_2 = \theta_2, \qquad \omega_3 = \theta_3 + \dot{\varphi}.$$

These may be used in any of the equations *except* (4), p. 38.

The velocities of the body may also be expressed in terms of

$$\omega_x,\ \omega_y,\ \omega_z \quad \textit{along} \quad X,\ Y,\ Z,$$

which specify the same motion as ω_1, ω_2, ω_3, and are therefore the X, Y, Z projections of ω_1, ω_2, ω_3; hence

(3) $$\begin{cases} \omega_x = -\dot{\psi}\sin\theta\cos\varphi + \dot{\theta}\sin\varphi, \\ \omega_y = \dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi, \\ \omega_z = \dot{\psi}\cos\theta + \dot{\varphi}. \end{cases}$$

These are to be used in (4), p. 38, because they refer to axes *fixed in the body;* the subscripts are inconsistent but this is not easy to avoid without making the notation cumbersome. When $\varphi = 0$, (2) and (3) give the same result.

Ex. 98. Prove that ω_1, ω_2, ω_3 are the projections of ω_x, ω_y, ω_z on KHZ.

42. Gyro or Top with a Fixed Point. Let a gyro of revolution of which A is the moment of inertia about every axis through the origin and perpendicular to Z, Fig. 38, be pivoted at a fixed point O with its centroid on the axle and at l above O. With the notation of §§ 20, 22, the moments about K, H, Z are

$$L = 0, \qquad M = wl \sin \theta, \qquad N = 0,$$

where w is the weight of the gyro.

The angular momenta about K, H, Z are

$$h_1 = A\omega_1, \qquad h_2 = A\omega_2, \qquad h_3 = C\omega_3,$$

or from (1), (2),

$$h_1 = -A\dot{\psi} \sin \theta, \qquad h_2 = A\dot{\theta}, \qquad h_3 = C(\dot{\psi} \cos \theta + \dot{\varphi}).$$

When these are substituted in (1), p. 32, C will be constant because its axis is fixed in the body, and A will be constant because, although its axis moves in the body, the gyro is a solid of revolution. Equations (3), p. 32, are necessary for a body not having this symmetry about planes through Z. The results of the substitution are easily found to be

$$(4) \qquad \begin{cases} A\ddot{\psi} \sin \theta + 2A\dot{\theta}\dot{\psi} \cos \theta - Cn\dot{\theta} = 0, \\ A\ddot{\theta} + Cn\dot{\psi} \sin \theta - A\dot{\psi}^2 \sin \theta \cos \theta = wl \sin \theta, \end{cases}$$

where n is the initial and constant value of ω_3.[1]

Ex. 99. Show that the equation of moments about OV is

$$\frac{d}{dt}(h_3 \cos \theta - h_1 \sin \theta) = 0.$$

This may be shown in two ways: by projecting the expression for L, M, N on OV or, better, by noticing that since V is fixed in direction the angular momentum about it can suffer only scalar (magnitude) changes.

43. The Integrals. It is stated in the text just preceding Ex. 99 that ω_3 remains constant. It might seem as if the constancy of ω_3 were due solely to the absence of applied moments about Z. But it is not: it depends also on the equality of angular momenta about K, H. There are no moments about K, yet it will be

[1] See first paragraph, § 43.

found that ω_1 is not constant. The absence of applied moments means that there is no *vector* change of momentum, but the scalar change need not be—and generally is not—zero. However, the angular momentum about the *fixed* axis V, about which there are no moments, must be constant, say H. From Ex. 99, with the values of h_1, h_3 replaced by those given above,

(1) $$Cn \cos \theta + A\dot{\psi} \sin^2 \theta = H,$$

which may also be got from first principles.

This is one integral of (4), p. 77. Another is derived from the conservation of energy, for if K is the kinetic energy plus the potential energy $wl \cos \theta$, we have

$$\tfrac{1}{2}(A\omega_1^2 + A\omega_2^2 + Cn^2) + wl \cos \theta = K$$

or

(2) $$A(\dot{\psi}^2 \sin^2 \theta + \dot{\theta}^2) = k - 2wl \cos \theta,$$

where

$$k = 2K - Cn^2.$$

Equations (1), (2), and

(3) $$\omega_3 = \dot{\psi} \cos \theta + \dot{\varphi} = n$$

can be used to find three of the four quantities θ, ψ, φ, t.

For brevity put

$$\cos \theta = u, \qquad wl = M, \qquad Cn = N.$$

With this notation the result of eliminating $\dot{\psi}$ from (1) and (2) is

(4) $$\begin{aligned} A^2\dot{u}^2 &= A(k-2Mu)(1-u^2)-(H-Nu)^2 \\ &= 2AMu^3-(N^2+Ak)u^2+2(HN-AM)u+Ak-H^2 \\ &= U. \end{aligned}$$

It follows from (4) that the time taken by the axle in moving from its initial angle $\theta_0 = \arccos u_0$ to any other position $\theta = \arccos u$, is

(5) $$t = A\int_{u_0}^{u} \frac{du}{\sqrt{U}},$$

in which the double sign $\pm$ is to be understood before the radical.

The elimination of dt from (1), (4) gives

$$(6) \qquad \psi - \psi_0 = \int_{u_0}^{u} \left(\frac{H - Nu}{1 - u^2} \right) \frac{du}{\sqrt{U}}.$$

From (3),

$$ndt = ud\psi + d\varphi.$$

Therefore from (6),

$$(7) \qquad \varphi - \varphi_0 = nt - \int_{u_0}^{u} \left(\frac{H - Nu}{1 - u^2} \right) \frac{udu}{\sqrt{U}}.$$

Equations (5)–(7) were found by Lagrange in 1788. Equation (7) can be changed to type (6) by a transformation due to Klein and Sommerfeld:[1]

$$\begin{aligned} \left(\frac{Hu - Nu^2}{1 - u^2} \right) \frac{du}{\sqrt{U}} &= \left(\frac{Hu - Nu^2}{1 - u^2} \right) \frac{dt}{A} \\ &= \left(\frac{Hu - N + N - Nu^2}{1 - u^2} \right) \frac{dt}{A} \\ &= -\left(\frac{N - Hu}{1 - u^2} \right) \frac{du}{\sqrt{U}} + \frac{N}{A} dt, \end{aligned}$$

whence

$$(8) \qquad \varphi - \varphi_0 = \int_{u_0}^{u} \left(\frac{N - Hu}{1 - u^2} \right) \frac{du}{\sqrt{U}} + n \left(1 - \frac{C}{A} \right) t.$$

44. The Cubic. Equation (4) may be written in the form

$$U = A^2 \dot{u}^2 = A^2 \dot{\theta}^2 \sin^2 \theta.$$

Since $\dot{\theta} = 0$ when $U = 0$, even when $\theta = 0$, as will be seen presently, the nutation is zero at every root of the cubic $U = 0$.

Values of $u = \cos \theta$ that exceed unity and values that make U negative ($\dot{\theta}$ imaginary) have no physical meaning, but can be used in studying the properties of the equation $U = 0$, which is a cubic in u. The roots are located by the variation of signs except in the case of $H = N$, where it is found that one root lies between

[1] *Theorie des Kreisels*, II, p. 223.

$u = 1$ and $u = \infty$ because, as the reader may verify, $dU/du < 0$ at $u = 1$. The values of U are plotted in Fig. 42 for $H > N$.

$H \neq N$, $\dot{\theta}$ real			$H = N$, $\dot{\theta}$ real		
u	U	Number of roots	u	U	Number of roots
$+\infty$	$+\infty$		$+\infty$	$+\infty$	
$+1$	$-(H-N)^2$	one	$+1$	0	one
0	$A^2\dot{\theta}^2$	one	0	$A^2\dot{\theta}^2$	one
-1	$-(H+N)^2$	one	-1	$-4N^2$	one
$-\infty$	$-\infty$		$-\infty$	$-\infty$	

$H \neq N$. If $\dot{\theta}$ is imaginary at $u = 0$, then $U < 1$ and there are *either* two roots between $u = 0$ and $u = +1$, *or* two roots between $u = 0$ and $u = -1$. In the following we shall take only $\theta \leqq 90°$.

When axle passes through $u = 0$ (top horizontal), the nutation changes sign (θ alternately decreasing and increasing) and passes through zero. Since $U \neq 0$ and therefore $\dot{\theta} \neq 0$ at $u = 1$ when $H \neq N$, the top cannot become *exactly* vertical in this case. However if H and N differ only infinitesimally the axle can approach infinitesimally close to the vertical.

$H = N$. As $\dot{\theta} = 0$ at $u = +1$ but not at $u = -1$, except in the extreme case of no spin and thus no top, the axle might rise to the upward vertical but cannot fall to the downward vertical.

Let the top, spinning vertically upright, be given a downward swing; as this cannot produce angular momentum about the vertical, H must equal N. But $\dot{\theta} = 0$ at $u = 1$ when $H = N$. Consequently, since the swing cannot be regarded as a nutation, it must be a precession. Now from (1), p. 78,

$$2A\dot{\psi} = N \quad \text{at} \quad u = 1,$$

which seems to mean that all tops that swing through the vertical, do so with the same precession regardless of their initial

motion. The difficulty is that we are dealing with a discontinuity: the plot of $\dot{\psi}$ in Fig. 39 shows an abrupt change from $+\infty$ to $-\infty$ as H passes through the value N; see also § 45.

45. Periodicity. The table in § 43 shows that there are always two values of $u = \cos\theta$, the roots of $U = 0$, at which $\dot{\theta} = 0$. If they are $u_1 = \cos\theta_1$, $u_2 = \cos\theta_2$, it is evident from Fig. 42 that θ must lie between θ_1, θ_2, for otherwise we should have the physically impossible case of $U < 0$ and imaginary nutation.

Imagine a sphere of any radius r drawn about O in Fig. 38. The piercing point of the axle—call it the apex of the gyro—will move between two limiting circles of latitude at θ_1, θ_2. The linear velocity of the apex, $\pm r(\dot{\psi}^2 \sin^2\theta + \dot{\theta}^2)^{1/2}$, is independent of ψ and has always the same value at the same θ: (2), p. 78. Moreover it is inclined to the meridian at arctan $\dot{\psi}\sin\theta/\dot{\theta}$ which is also independent of ψ. Consequently the locus of the apex always crosses a given circle of latitude at the same angle, the angle changing sign with $\dot{\theta}$, and consists of a series of repeated—but not retraced—migrations, § 46, No. 11. In short, the motion is periodic.

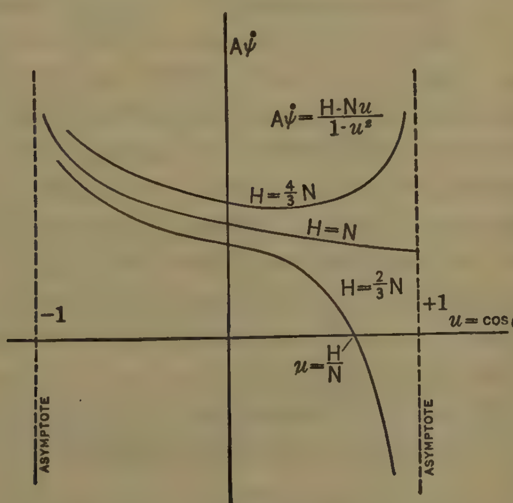
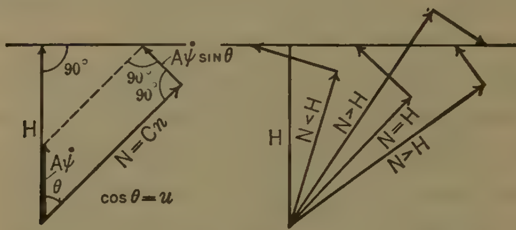


FIG. 39

46. The Precession. From (1), p. 78, we have

$$A\dot{\psi} = \frac{H - Nu}{(1 - u^2)},$$

which is plotted in Fig. 39.



I II

FIG. 40

Figure 40 is the vector representation of (1); in these diagrams θ is measured from the vertical H and is positive for all positions of the axle. Figure i shows the obvious relations among the several quantities; note in ii that $\dot{\psi}$ cannot be negative unless $N > H$.

(*a*) $H \geq N$. The precession cannot be zero or negative.

(*b*) $H < N$. The curve in Fig. 39 cuts the u-axis at H/N. It will now be proved that *both* roots u_1, u_2, at which $\dot{\theta} = 0$, cannot exceed H/N, *i.e.*, $\dot{\psi}$ cannot be negative at both boundary circles of latitude. For Fig. 39, $H < N$, shows that $\dot{\psi}$ is positive at the lower circle (smaller u, larger θ) if it vanishes at the upper. Now the precession is always negative if both roots lie beyond H/N. But from the second equation of (4), p. 78, $\ddot{\theta}$ is positive when $\dot{\psi}$ is negative. This is impossible in periodic motion, since it requires $\dot{\theta}$ to be always increasing. Hence the precession cannot be negative at both circles.

(*c*) Consider now the consequences of zero precession at the lower circle: this would require it to be negative at the upper. But for

$$u = u_1, \quad \dot{\psi} = 0, \quad \dot{\theta} = 0, \quad \text{and} \quad u = u_2, \quad \dot{\psi} = \dot{\psi}_2, \quad \dot{\theta} = 0,$$

(2), p. 78, gives

$$k - 2Mu_2 - (k - 2Mu_1) = A(\dot\psi_2^2 \sin^2 \theta - 0);$$

whence $2M(u_1 - u_2)$ is positive and

$$u_1 > u_2,$$

which is absurd since u_1 refers to the lower circle and is the smaller root. The precession can thus be zero only at the *upper* circle. The physical interpretation of this is that if the kinetic energy is $\frac{1}{2}Cn^2$ at the lower circle, where $\dot\theta = 0$ and $\dot\psi$ is *assumed* zero, it will be greater by the energy of precession or nutation when the axle rises, which is impossible because no work is done *on* the system.

(*d*) When $\dot\psi = 0$ the locus is evidently tangent to a meridian circle unless there is a point of inflexion. Since $\dot\psi$ must change sign as it passes through zero, there can be no point of inflexion tangent to a meridian. When $\dot\theta = 0$ and $\dot\psi \neq 0$ the locus of the apex is tangent to one of the boundary circles.

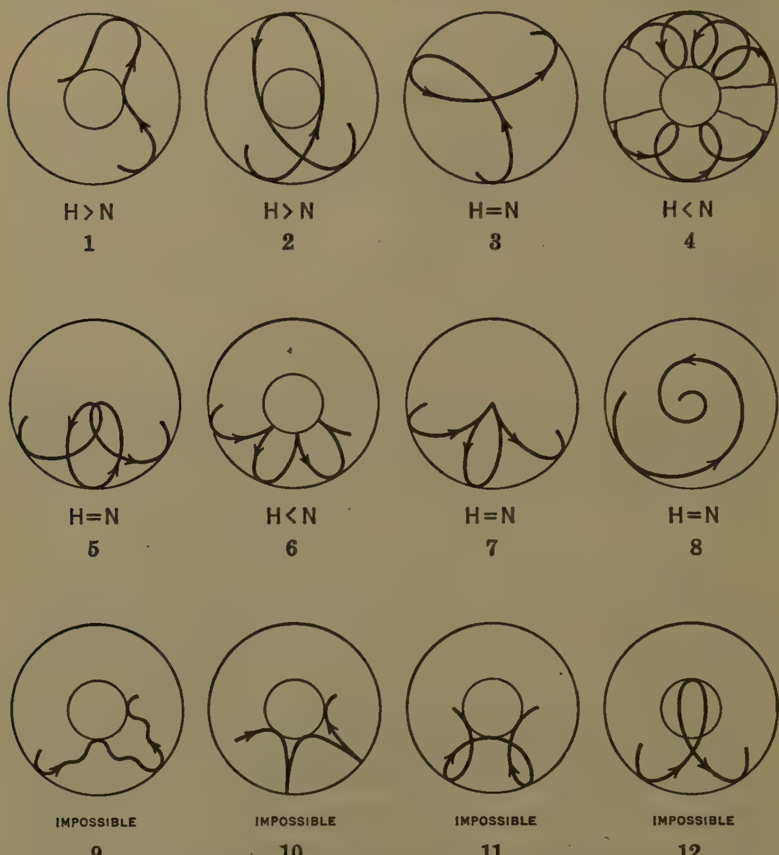
Ex. 100. Show that double points cannot exist at either circle.

47. Locus of the Apex. Typical forms of loci[1] of the apex—some possible, some not—are shown in Fig. 41 for positive $wl \sin \theta$ and $\theta < 90°$. These views are projections on the horizontal plane; n is counterclockwise as seen here, and $N = Cn$ is positive. Intermediate forms will suggest themselves; *e.g.* a somewhat star-shaped figure exists between No. 1 and No. 2. Differences in loci arise from differences in the constants of the top: A, C, n, wl, and the initial values of θ, $\dot\theta$, $\dot\psi$. They are interesting mathematically, but a few comments on the impossible cases will suffice for the engineer.

No. 9. If this type were possible it seems reasonable to believe that the constants of the top could be selected either to make $\dot\psi = 0$ at a radial tangent (a meridian of the sphere on

[1] Scale diagrams and the necessary elliptic function analysis are given in Greenhill, *Report on Gyroscopic Theory*, 1914; for stereoscopic diagrams see Greenhill and Dewar, *Engineering*, vol. 64, 1897, p. 311.

which the apex moves), which is impossible by Fig. 39, $H > N$, since $\dot{\psi} > 0$, or to have a point of inflection at a meridian, which contradicts (*d*), p. 83.



CENTROID ABOVE SUPPORT. SPIN LEFT-HANDED AS VIEWED FROM ABOVE

FIG. 41

No. 10. See (*c*), p. 82.

No. 11. By proper choice of the constants there would be cusps at the lower (larger) circle contrary to (*c*). Furthermore if cusps were possible there would be nothing to distinguish No. 11 from No. 10. Hence the axle must advance right-

handed (counterclockwise in Fig. 41) about the vertical in its motion between the boundary circles; this means that the integral in (6), p. 79, is positive [1] between u_1, u_2.

No. 12. The nutation is not zero at u_2.

48. Nearly Equal Roots. The time of one swing, a half-cycle, or half-period, from u_1 to u_2, is by (5), p. 78,

$$\frac{t}{A} = \int_{u_1}^{u_2} \frac{du}{\sqrt{U}},$$

where

$$U = 2AM(u - u_1)(u_2 - u)(u_3 - u),$$

$$u_1 < u < u_2 < 1 < u_3; \qquad u_1,\ u_2\ \textit{positive}.$$

The functions U and $U^{-1/2}$ are plotted in Fig. 42; the reciprocal curve $U^{-1/2}$ has asymptotes at u_1, u_2, and only the part between them is shown.

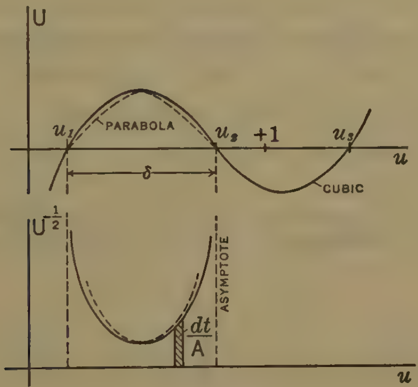


FIG. 42

Since the element of time is given by the strip of area

$$U^{-1/2}du = \frac{dt}{A},$$

t/A is the area from u_1 to u_2 under the $U^{-1/2}$ curve. If t is small,

[1] Hadamard, Bulletin des sciences mathématiques, 1895, p. 228, gives a mathematical proof.

u_1 and u_2 lie close together and the cubic U between these points may be replaced by a parabola through u_1, u_2, and the middle ordinate at $u_1 + \delta/2$. The dotted curves are the parabola and its reciprocal curve. The lower curve shows that the approximate time will be somewhat *larger* than the true time if the parabola is on the concave side of the cubic.

The equation of the parabola is

$$P = 2AM(u - a)(b - u)c,$$

P being the ordinate for any abscissa u. For it to pass through u_1, $u_1 + \delta/2$, u_2, we must have

$$P = 0 \text{ at } u_1 \text{ and } u_2, \qquad P = U \text{ at } u_1 + \delta/2.$$

Hence

$$a = u_1, \qquad b = u_2, \qquad c = u_3 - u_1 - \delta/2$$

or

$$P = 2AM(u - u_1)(u_2 - u)(u_3 - u_1 - \delta/2).$$

At u_1 and u_2, the slopes of parabola and cubic are respectively $\pm\, 2AM(u_2 - u_1)(u_3 - u_1 - \delta/2)$, $\pm\, 2AM(u_2 - u_1)(u_3 - u_1)$, whence the parabola lies inside of the cubic.

The approximate time t_a is given by

$$\frac{t_a}{A} = \frac{1}{\sqrt{2AM(u_3 - u_1 - \delta/2)}} \int_{u_1}^{u_2} \frac{du}{\sqrt{(u - u_1)(u_2 - u)}}.$$

But [1]

$$\int_{u_1}^{u_2} \frac{du}{\sqrt{(u - u_1)(u_2 - u)}} = -\left[\arcsin \frac{-2u + u_2 + u_1}{u_1 - u_2}\right]_{u_1}^{u_2}.$$
$$= \pi.$$

When δ is negligible the time of a small swing is thus

$$t_a = \frac{\pi A}{\sqrt{2AM(u_3 - u_1)}}.$$

When u_1 and u_2 are approximately equal, the sum of the roots of the cubic, (4), p. 78, is $2u_1 + u_3$, which must be the coefficient

[1] Peirce, *Short Table of Integrals*, No. 113.

of u^2, with sign changed, when the coefficient of u^3 is unity, *i.e.*,

$$2u_1 + u_3 = \frac{N^2 + Ak}{2AM},$$

or

$$u_3 - u_1 = \frac{N^2 + Ak}{2AM} - 3u_1.$$

If $\dot\psi = \mu$ at u_1, u_2, where $\dot\theta = 0$, (2), p. 78, gives

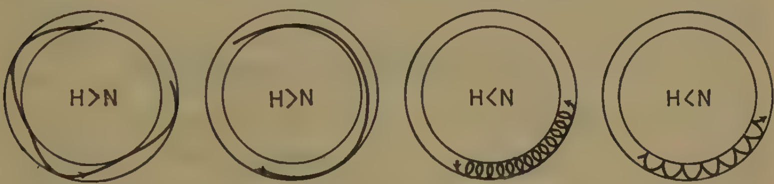
$$k = 2Mu_1 + A\mu^2(1 - u_1^2).$$

Hence

$$t_a = \frac{\pi A}{\sqrt{N^2 - 4AMu_1 + A^2(1 - u_1^2)\mu^2}},$$

where μ is the precession for *steady* motion as found from (4), p. 77, when $\dot\theta = 0$ and $\ddot\theta = 0$; see § 50.

When the limiting circles are close together, as they are for small δ or ϵ, the motion is *smooth*, Fig. 43, I, II, or *tremulous*, III, IV; these convenient terms are due to Routh.[1]



I II III IV

FIG. 43

Ex. 101. Show from t_a that the gyro cannot become vertical unless $C^2n^2 > 4wlA$.

Ex. 102. The last factor of P on p. 86 may be written $u_3 - u_2 + \epsilon/2$. Show that the approximate time is $\pi A/\sqrt{2AM(u_3 - u_2)}$ and is *smaller* than the true time.

49. Small Oscillations. The small oscillations about a state of steady motion between two close circles of latitude can be obtained directly from the differential equations themselves. This is the most instructive and indeed effective way of solving

[1] *Advanced Rigid Dynamics*, § 30.

the problem. Equations (4), p. 77, are

$$A\ddot{\psi}\sin\theta + 2A\dot{\theta}\dot{\psi}\cos\theta - N\dot{\theta} = 0,$$

$$A\ddot{\theta} + N\dot{\psi}\sin\theta - A\dot{\psi}^2\sin\theta\cos\theta = M\sin\theta,$$

where

$$N = Cn, \qquad M = wl.$$

If θ and $\dot{\psi}$ never vary much from their mean values θ_m and μ, we may put

(1) $$\theta = \theta_m + \vartheta, \qquad \dot{\psi} = \mu + \epsilon,$$

where ϑ, ϵ are so small that their products and squares are negligible. We may write, approximately,

$$\sin(\theta_m + \vartheta) = \sin\theta_m + \vartheta\cos\theta_m,$$

$$\cos(\theta_m + \vartheta) = \cos\theta_m - \vartheta\sin\theta_m,$$

$$\dot{\theta}\dot{\psi} = \dot{\vartheta}(\mu + \epsilon) = \dot{\vartheta}\mu,$$

and the equation for $\ddot{\psi}$ reduces to

$$A\dot{\epsilon}\sin\theta_m + 2A\dot{\vartheta}\mu\cos\theta_m - N\dot{\vartheta} = 0.$$

Hence

$$A\sin\theta_m\int_0^{\epsilon} d\epsilon = (N - 2A\mu\cos\theta_m)\int_0^{\vartheta} d\vartheta$$

or

(2) $$A\epsilon\sin\theta_m = (N - 2A\mu\cos\theta_m)\vartheta.$$

Since μ is the precession in steady motion, when $\theta = \theta_m$, $\dot{\theta} = 0$, and $\ddot{\theta} = 0$, the equation for $\ddot{\theta}$ gives

(3) $$M = N\mu - A\mu^2\cos\theta_m.$$

When M, θ, $\dot{\psi}$, $\cos\theta_m = u_m$, and ϵ are substituted in the $\ddot{\theta}$ equation and second order terms are rejected, there results

(4) $$A^2\ddot{\vartheta} = -\{N^2 - 4AMu_m + A(1 - u_m{}^2)\mu^2\}\vartheta.$$

If M is replaced by its value in terms of N and μ it is seen that the coefficient of ϑ is always positive provided $N^2 > 4AMu_m$. The motion is therefore simple harmonic, Fig. 43, I, II, the time of a cycle or period being that found on p. 87.

50. Steady Motion. Equation (3), p. 88, defines the steady motion (no nutation) of a top. Its roots are

$$\mu = \frac{N}{2Au}(1 \pm \Delta), \tag{1}$$

where

$$\Delta = \left(1 - \frac{4AMu}{N^2}\right)^{1/2};$$

μ is positive and finite for $u > 0$, and real when

$$N^2 \geq 4AMu. \tag{2}$$

There are thus two precessions—a fast and a slow—for every inclination of the axis, provided the spin (n in $N = Cn$) is large enough. If condition (2) is satisfied, (4), p. 88, shows that a slight nutation will produce simple harmonic oscillations.

Ex. 103. The bracket in (4), p. 88, is positive when $A(1 - u^2)\mu^2 > 4AMu$. Why is this condition neither necessary nor sufficient for harmonic oscillations?

Ex. 104. Show that the oscillations are faster for the fast precession.

Ex. 105. In elementary discussions of the gyroscope it is often stated that the moment is *the* cause of the precession "as when a force acts on a body at right angles to the direction of its linear motion." Show that the fast precession grows smaller when M increases. Is M the sole cause of precession?

There is a peculiar difference in the behavior of the axle for the two precessions. From (1)

$$2Au\mu - N = \pm N\Delta;$$

hence from (2), p. 88,

$$A\epsilon \sin \theta_m = \mp N\vartheta\Delta,$$

the upper sign referring to the fast (large) μ. Since ϵ and ϑ are opposite in sign for the fast μ, the axle falls when μ increases. For the same reason, the axle rises when the slow (small) μ increases. This case—but only this—corresponds to Kelvin's rule:[1] "Hurry on the precession, and the body rises in opposition to gravity."

[1] Perry, *Spinning Tops*, p. 70. See Greenhill, *Report on Gyroscopic Theory*, Chap. I, §§ 5, 6, for the result of changing H in (1), p. 78.

Ex. 106. Derive these results from $d\mu/du$.

Ex. 107. Show that $\ddot{\theta}$ is positive when $\dot{\psi} >$ fast μ and $\ddot{\theta}$ is negative when $\dot{\psi} >$ slow μ.

The limiting case of steady motion occurs when the spin dies down until

$$N^2 = 4AMu, \quad \text{and} \quad \Delta = 0;$$

the axle has now reached its greatest rise. The two precessions are

$$\mu = \frac{N}{2Au}, \quad 0.$$

When $\mu = 0$, the axle starts to fall and the motion is tremulous as in Fig. 43, IV; the motion is also unstable at the fast μ.

N is large for a rapid spin—the usual case—and (1) expanded by the binomial theorem gives, when small terms are neglected,

$$\mu = \frac{N}{Au}, \frac{M}{N}, \textit{ approximately},$$

which are also the same as for a gyro supported near its centroid (l small in $M = wl$). Notice that the fast μ is independent of M, approximately.

The fast μ is of the order of magnitude of n, and is infinite at $u = 0$, $(\theta = 90°)$; the slow μ is very slow and is zero when $M = 0$, which corresponds to the fact that a gyro axle may maintain a fixed direction when supported at the centroid. On the other hand, the fast precession once imparted will be maintained, but no other precession is compatible with steady motion.

The analysis has shown that both precessions are possible. The fast precession is more of the nature of a spin and must be imparted initially; it will not result from the action of the moment $Mu = wl \cos \theta$ when the axle is released from rest, because the work of the moment is generally not large enough to produce the precessional kinetic energy. It is, however, often conspicuous in the diabolo (double cone, vertex to vertex,

spun in the bight of a string). The difference between the two precessions seems to be partly a matter of stability. From the values of μ just found, we have

$$\frac{d\mu}{du} = -\frac{N}{Au^2}, 0, \textit{approximately}.$$

whence a slight nutation produces a more violent disturbance of the fast than of the slow precession.

Ex. 108. Show that only the slow precession follows Huntington's Rule, p. 49.

51. The Upright Top. It is evident from (2), p. 89, that a top cannot become upright or sleep, $u = 1$, unless

(1) $$N^2 \geqq 4AM.$$

Such a top is called *strong* by Klein and Sommerfeld. A *weak* top cannot rise higher than

$$u = \frac{N^2}{4AM},$$

which is less than unity if $N^2 < 4AM$. This latter condition does not exclude $N^2 > 4AMu$: it merely sets an upper limit to u.

Ex. 109. The top cannot rise if it is too heavy, *i.e.* the kinetic energy of precession, which is responsible for the rise, must in general exceed the increase of potential energy. Show therefore that $\mu < 2M/N$ and that this is equivalent to $N^2 > 4AM$.

52. Reaction of the Support. The forces exerted on the gyro at the point O in Fig. 38 can be found from the accelerations of the centroid. The accelerations are produced by scalar and magnitude changes of velocity according to Theorem III, p. 4. The velocity of the centroid, l above O, has components $l\dot{\theta}$ perpendicular to l in the plane ZV, and $l\dot{\psi}\sin\theta$ parallel to $+H$. Their scalar rates of change are in the directions of the velocities. The directional changes are caused by the rotation of $l\dot{\theta}$ at $\dot{\theta}$, $l\dot{\theta}\cos\theta$ at $\dot{\psi}$, and $l\dot{\psi}\sin\theta$ at $\dot{\psi}$. Hence if the forces at O are V upward, H parallel to $+H$, and P perpendicular to both V

and H, toward the reader,

$$w - V = m(l\ddot{\theta}\sin\theta + l\dot{\theta}^2\cos\theta),$$
$$H = m(l\ddot{\psi}\sin\theta + 2l\dot{\psi}\dot{\theta}\cos\theta),$$
$$P = m(l\ddot{\theta}\cos\theta - l\dot{\theta}^2\sin\theta - l\dot{\psi}^2\sin\theta).$$

The forces parallel to the principal axes X, Y, Z can be got in a somewhat different way. From Ex. 38, p. 21,

$$\ddot{x} = z\dot{\omega}_2 - y\dot{\omega}_3 + \omega_2(\omega_1 y - \omega_2 x) - \omega_3(\omega_3 x - \omega_1 z).$$

Substituting $\dot{\omega}_1$, $\dot{\omega}_2$, $\dot{\omega}_3$ from (4), p. 38, multiplying by the mass m of *one* particle, and summing for the whole body, we get

$$\Sigma m\ddot{x} = \frac{M + (C - A)\omega_1\omega_3}{B}\Sigma mz - \frac{N + (A - B)\omega_1\omega_2}{C}\Sigma my$$
$$+ \omega_1\omega_2\Sigma my + \omega_1\omega_3\Sigma mz - (\omega_2^2 + \omega_3^2)\Sigma mx,$$

with similar results for the other axes.

If X, Y, Z are the reactions along X, Y, Z, we have

$$X + w\sin\theta\cos\varphi = \Sigma m\ddot{x},$$
$$Y - w\sin\varphi\sin\varphi = \Sigma m\ddot{y},$$
$$Z - w\cos\theta = \Sigma m\ddot{z},$$

where the components of w are found by first resolving it along Z and K. Since the centroid is on Z, we have

$$\Sigma mx = 0, \qquad \Sigma my = 0, \qquad \Sigma mz = \frac{w}{g}l,$$
$$A = B, \qquad M = wl\sin\theta, \qquad L = 0, \qquad N = 0.$$

Hence

$$X + w\sin\theta\cos\varphi = \frac{wl\sin\theta + (C - A)\omega_1\omega_3}{B}\,\frac{w}{g}\,l + \frac{w}{g}\,l\omega_1\omega_2,$$

$$Y - w\sin\theta\sin\varphi = \frac{(C - A)}{A}\,\frac{w}{g}\,l\omega_2\omega_3 + \frac{w}{g}\,l\omega_2\omega_3,$$

$$Z - w\cos\theta = -(\omega_1^2 + \omega_2^2)\frac{w}{g}\,l.$$

Ex. 110. Can an unsymmetric gyro, with A, B, C all different, precess at constant speed and inclination?

CHAPTER VII

MOTION ON A PLANE

53. Disk: General Equations. The general problem of a body rolling or spinning on a plane is of somewhat more mathematical than engineering interest. A few simple cases will be discussed here in order to show the method of attack and to bring out the meaning of kinetic stability. The steady motion of a wheel or disk was dealt with in § 29, but the motion was not shown to be stable. The criterion of stability is that small perturbations of the steady motion shall produce oscillations of small amplitude. In deriving the general equations of motion, it might seem better to put the origin of coordinates at the point of contact in Fig. 44 in order to eliminate the ground reactions. This is not permissible in the equations of § 22 because the origin must be *fixed on the body:* if it is not, the velocities in (3), p. 32, are not common to all the particles and cannot be put outside of the summation signs. An exception occurs when the point of contact lies on the instantaneous axis, which is the case in Fig. 44 when there is no slipping.

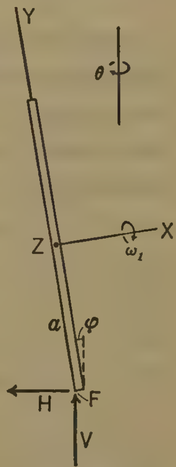


FIG. 44

We shall take the origin at the centroid because the ground reactions are important enough to be considered. Let XYZ be principal axes moving with the disk. The XY plane stays vertical but turns with the disk at the rate θ about a vertical line, and at $\dot{\varphi}$ about a horizontal line. Unless the ground is sufficiently rough to exert a reaction H there will be no horizontal acceleration of the centroid in the XY plane and the disk will fall. Another

friction force F will act parallel to Z; assume it to point into the paper. The resultant of F and H cannot exceed μV, where μ is the coefficient of friction.

The values of the applied moments about X, Y, Z are easily found from the angular momenta

$$A\omega_1, \quad -B\theta\cos\varphi, \quad B\dot{\varphi},$$

by means of Theorem X, p. 33, or by substitution in (1), p. 32, or (4), p. 38. The equations of rotation are

$$(1) \qquad Fa = A\dot{\omega}_1,$$

$$(2) \qquad B\dot{\theta}\cos\varphi = A\omega_1\dot{\varphi} + 2B\dot{\varphi}\theta\sin\varphi,$$

$$(3) \qquad Va\sin\varphi - Ha\cos\varphi = B\ddot{\varphi} + A\omega_1\theta\cos\varphi + B\theta^2\sin\varphi\cos\varphi.$$

Equation (2) shows that $\dot{\theta}$ is positive when $\dot{\varphi}$ is positive, which means that the falling of the disk, due to the moment of its weight about F, will increase the vertical spin. This is why the path traced by the point of contact increases in curvature as the disk falls. The moment of the weight must ultimately be responsible for the gross change of φ; nevertheless the *motion* $\dot{\varphi}$ and *not the moment* of the weight turns or steers the disk. For if $\dot{\varphi}$ is negative, $\dot{\theta}$ will be negative and the path of contact will get flatter and may change curvature as in Fig. 45. The disk was about the size of the last loop and was started with a wobble; the experiment may be made with a silver dollar on a freshly ironed tablecloth. The original curve was several times as large. The reactions are eliminated from (1)–(3) by means of the force equations which are obtained as follows.

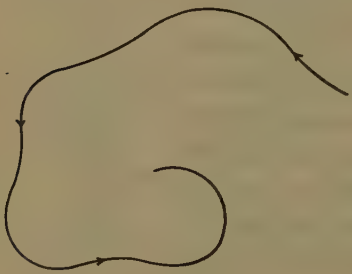


FIG. 45

If there is no slip the point of contact is the instantaneous center. Any slip produced in giving too much push for the

spin θ, or too much spin for the push, soon vanishes on account of friction. The velocities of the centroid are then

$$a\omega_1 \ \ along + Z, \qquad a\dot{\varphi} \ \ along - X.$$

Their changes of magnitude and direction give rise to accelerations parallel to F, V, H as contained in the equations

(4) $$-F = m(a\dot{\omega}_1 - a\dot{\varphi}\theta \cos\varphi),$$

(5) $$V - mg = -ma\ddot{\varphi}\sin\varphi - ma\dot{\varphi}^2\cos\varphi$$

(6) $$H = m(a\omega_1\theta + a\ddot{\varphi}\cos\varphi) - ma\dot{\varphi}^2\sin\varphi$$

Equations (1) and (4) show that F cannot vanish unless $\dot{\varphi}$ is zero, a result that cannot be anticipated without careful consideration of all the circumstances of the motion. Since the differential equations obtained by eliminating the reactions are not solvable by elementary methods we shall take up only special cases.

Ex. 111. Eliminate F, V, H from (1)–(6) and show that the same result is got by taking the origin at the ground.

54. Rolling in a Straight Line. If the disk in Fig. 44 is placed in a perfectly vertical plane, it may be rolled in a straight line at any speed; it may indeed be made to stand in a vertical plane. But there will be instability in each case, although in the former the motion may be stabilized by a sufficiently large velocity of rolling. Its stability depends on the character of the motion produced by a small disturbance. Suppose then that the disk is started vertically with a small nutation $\dot{\varphi} = \nu$, but with $\theta = 0$. The precession θ will therefore be small provided φ is small. Terms of the second order, θ^2, $\theta\dot{\varphi}$, etc., will be negligible and $\cos\varphi = 1$, $\sin\varphi = \varphi$. With these values (1)–(6) become

(7) $$Fa = A\dot{\omega}_1,$$

(8) $$B\dot{\theta} = A\omega_1\dot{\varphi},$$

(9) $$Va\varphi - Ha = B\ddot{\varphi} + A\omega_1\theta,$$

(10) $$-F = ma\dot{\omega}_1,$$

(11) $$V = mg,$$

(12) $$H = ma(\ddot{\varphi} + \omega_1\theta).$$

From (7) and (10), we find

$$(A + ma^2)\dot{\omega}_1 = 0,$$

or

$$\omega_1 = n,$$

which is constant to terms of the first order.

From (8),

$$B\theta = An\varphi + (c = 0), \tag{13}$$

where $c = 0$ because $\theta = 0$ at $\varphi = 0$. Of course $\theta = 0$ at $\varphi = 0$ does not imply $\dot{\theta} = 0$; in fact $B\dot{\theta} = An\nu$ at $\varphi = 0$ since ν was assumed as the initial value of $\dot{\varphi}$.

From (9), (11), (12), (13)

$$(B + ma^2)\ddot{\varphi} = -\left\{\frac{(A + ma^2)An^2}{B} - mag\right\}\varphi. \tag{14}$$

This motion is harmonic provided

$$A(A + ma^2)n^2 > maBg.$$

Equation (13) shows that θ varies as φ; this makes the precession oscillatory with the period of the nutation. Consequently the disk rolls in a sinuous or wavy line, with the motion smooth rather than tremulous. At points of inflection θ changes sign and passes through zero, whence, by (13), $\varphi = 0$ and the disk is upright. At other points of the path it leans toward the concave side.

The shimmy or wobble of the front wheels of a motor car resembles the motion of the disk. Shimmy is started by unequal compressions of the front tires, especially in the case of balloon tires, produced by unevenness in the roadway; this causes the front axle to tramp (oscillate about an axis parallel to the path) and the tramping induces precession which, if the periods are right, reinforces it.[1]

Ex. 112. Find the least n for stability of a disk, a hoop, and a hoop with a small heavy center.

Ex. 113. Find the period and length of a wave in Ex. 112.

[1] See also Huebotter, *Mechanics of front-wheel shimmy*, Journal of the Society of Automotive Engineers, April, 1927, pp. 423–425.

55. Disk Spinning Upright. When a slight disturbance is imparted to a disk spinning about a fixed point of tangency, with its plane vertical, it will roll in a circle of small radius; ω_1 and φ will be small but θ will be large. Eq. (1)–(6), § 53, to terms of the first order, are

(15) $$Fa = A\dot{\omega}_1,$$

(16) $$B\dot{\theta} = 0,$$

(17) $$Va\varphi - Ha = B\ddot{\varphi} + A\omega_1\theta + B\theta^2\varphi,$$

(18) $$-F = ma(\dot{\omega}_1 - \theta^2\varphi),$$

(19) $$V = mg,$$

(20) $$H = ma(\omega_1\theta + \ddot{\varphi}).$$

From (16), θ is constant to terms of the first order.

From (15) and (18),

$$(A + ma^2)\dot{\omega}_1 = ma^2\theta\dot{\varphi},$$

or as θ is constant,

(21) $$(A + ma^2)\omega_1 = ma^2\theta\varphi + (c = 0),$$

where the constant of integration is zero because the disk is started with its plane vertical and without any ω_1; the disturbance is caused by a small $\dot{\varphi}$.

From (17), (19), (20), and (21),

$$(B + ma^2)\ddot{\varphi} = -\{(B + ma^2)\theta^2 - mag\}\varphi,$$

the condition for stable motion being

$$(B + ma^2)\theta^2 > mag.$$

Pivot friction and air resistance, the latter being large for a spinning disk, have been disregarded; they slow down the spin but do not otherwise change the character of the motion.

Ex. 114. Find the least value of θ for the stability of a disk, a hoop, and a hoop with a small heavy center.

Ex. 115. Find the period of the φ motion in Ex. 114.

56. Disk nearly Horizontal. When a spinning coin dies down to an almost horizontal position, it rolls in what is nearly enough a circle of about its own diameter. As φ is not quite 90° we may put

$$\varphi = \varphi_0 + \nu,$$

where φ_0 is slightly less than 90° and ν is very small. In Fig. 27 ψ is almost 90°, whence ω_1 is small but θ is large. Approximately

$$\cos\varphi = \cos\varphi_0 - \nu, \qquad \sin\varphi = 1.$$

With $\theta = \theta_0 + \epsilon$, ϵ being small and θ_0 constant, (1)–(6), § 53, become, to terms of the first order,

(1) $$Fa = A\dot{\omega}_1,$$

(2) $$B\dot{\epsilon}\cos\varphi_0 = 2B\theta_0\dot{\nu},$$

(3) $$Va - Ha(\cos\varphi_0 - \nu) = B\ddot{\nu} + B\theta_0^2(\cos\varphi_0 - \nu),$$

(4) $$-F = ma\dot{\omega}_1,$$

(5) $$V - mg = -ma\ddot{\nu},$$

(6) $$H = ma\omega_1\theta_0.$$

From (1) and (4),

$$(A + ma^2)\dot{\omega}_1 = 0, \quad \text{or} \quad \omega_1 = n, \textit{ constant.}$$

Substituting from (5) and (6) into (3), we get

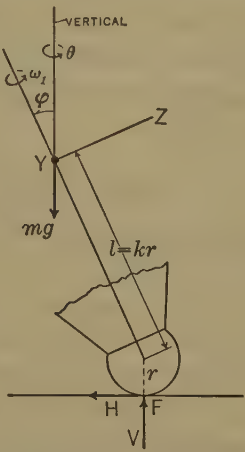
$$(B + ma^2)\ddot{\nu} = mag - (ma^2n\theta_0 + B\theta_0^2)\cos\varphi_0 + (B\theta_0^2 + ma^2n\theta_0)\nu.$$

Hence the motion is not oscillatory, and ν, and thus θ, increase with t. The increasing of θ can be heard in the rising note of a coin spinning on a plate just before it comes to rest.

57. Top on a Rough Plane. In order to simplify the mathematical analysis, we shall assume that the top—any solid of revolution—has a spherical peg or end. The results will serve to explain in a general way why some tops rise more readily than others and why, for instance, an egg (hard-boiled to prevent fluid friction, or a darning egg) will spin better on its small than its

large end. The first satisfactory explanation of the behavior of a top was given by Gallop,[1] who showed that dissipation of energy through friction is a necessary feature of the rise. Gallop's ideas, with some modification of detail, are presented below.

The friction reaction of the plane on the peg of the top in Fig. 46 is resolved into H toward the left and F perpendicular to the plane of the paper — the XZ plane. At present no assumption is made as to whether the friction is limiting.



Y POINTS TOWARD THE READER AND REMAINS HORIZONTAL; F IS PARALLEL TO Y INTO THE PAPER

FIG. 46

The axes move so that Y, perpendicular to the paper, stays horizontal, and the XZ plane vertical. Their angular velocities are

$$\theta_1 = \theta \cos \varphi, \quad \theta_2 = \dot{\varphi}, \quad \theta_3 = \theta \sin \varphi;$$

the angular momenta of the top are

$$A\omega_1, \quad B\dot{\varphi}, \quad B\theta \sin \varphi.$$

The change of angular momentum about, *i.e.*, projected on, the vertical, is merely scalar, d/dt, because its direction is constant, and is produced by the moment of F; hence

(1) $$\frac{d}{dt}(A\omega_1 \cos \varphi + B\theta \sin^2 \varphi) = Fl \sin \varphi,$$

from which F is to be eliminated by means of

(2) $$-Fr \sin \varphi = A\dot{\omega}_1.$$

Ex. 116. Show that (2) is not true unless the top is a solid of revolution.

[1] *The rise of a spinning top*, Transactions of the Cambridge Mathematical Society, vol. 19, 1903.

With the abbreviation $l = kr$, we get from (1) and (2)

$$A(k + \cos\varphi)\omega_1 + B\dot{\theta}\sin^2\varphi = K, \textit{ constant.} \tag{3}$$

This integral is named after Jellett, who obtained it in 1872. The simple derivation given here—which shows it to be entirely independent of the friction—is due to Routh.[1]

From (3),

$$d\varphi = \frac{A(k + \cos\varphi)d\omega_1 + Bd\dot{\theta}\sin^2\varphi}{A\omega_1\sin\varphi - B\dot{\theta}\sin\varphi\cos\varphi}.$$

Therefore, from (2),

$$\dot{\varphi} = -\frac{(k + \cos\varphi)Fr + B\dot{\theta}\sin\varphi}{A\omega_1 - B\dot{\theta}\cos\varphi}. \tag{4}$$

For a fast top, ω_1 is large, and the denominator of (4) is positive for $A > B$; when k, F, and r are large and φ is rather small, the numerator will also be positive, thus making $\dot{\varphi}$ negative. Under these circumstances the top will certainly rise.

When the top is only slightly inclined to the vertical, the second term in the numerator of (4) is small and $\dot{\varphi}$ may be taken as directly proportional to F. This means that when F acts in the sense of the precession $\dot{\theta}$ as assumed in Fig. 46, friction may be regarded as producing the rise of a top. This is the explanation usually given on the basis of the rule: hurrying the precession produces a rise (see p. 89). However, F does not necessarily act in the sense of $\dot{\theta}$ unless there is slip. When there is slip a large F may dissipate energy at too great a rate and in this way prevent the top from rising; see § 59.

When r is small, $\dot{\varphi}$ is small, and the time of rise will be correspondingly long, possibly so long as to permit air resistance and sliding friction to decrease the spin below the minimum allowable for rise. Whip-tops, which should rise quickly, should therefore have large well-rounded pegs and high centroids (large k).

Ex. 117. Explain why it is difficult to spin a long pencil on its point and why it becomes easier when the pencil is stuck through a disk of cardboard.

[1] *Advanced Rigid Dynamics*, 1905, § 243.

58. Least Spin. Let a top be started at φ_0 with spin ω_0 but no $\dot{\theta}$. Then if ν is its spin in the vertical position, $\varphi = 0$, Jellett's integral, (3), p. 100, gives

$$A(k + 1)\nu = K = A(k + \cos \varphi_0)\omega_0,$$

or

$$\omega_0 = \frac{(k + 1)\nu}{k + \cos \varphi_0},$$

from which it follows that ω_0 must certainly be greater than ν. In the rise from $\varphi = \varphi_0$ to $\varphi = 0$, the initial kinetic energy $\frac{1}{2}A\omega_0^2$ is reduced to its final value $\frac{1}{2}A\nu^2$ by the dissipated work of friction and the work of lifting, $wl(1 - \cos \varphi_0)$. Hence, neglecting the relatively small energy of precession and nutation,

$$\tfrac{1}{2}A\omega_0^2 > \tfrac{1}{2}A\nu^2 + wl(1 - \cos \varphi_0).$$

Replacing ν in terms of ω_0, we have

$$\frac{A\omega_0^2}{2} > \frac{wl(k + 1)}{1 + 2k + \cos \varphi_0}.$$

This sets a lower limit to ω_0 and shows that a top with a high centroid, *ceteris paribus*, requires a large spin.

Ex. 118. Show that in steady motion with pure rolling, the velocity of the centroid in Fig. 46 is

$$v = (l + r \cos \varphi)\omega_3 - r\omega_1 \sin \varphi, \quad \text{where} \quad \omega_3 = \dot{\theta} \sin \varphi.$$

59. Gallop's Criterion. The point of view taken by Gallop in regard to steady spin in the vertical position is the following:

"The energy in the steady motion with the axis vertical cannot in general be equal to the initial energy, so this state of motion could not be attained on perfectly rough or perfectly smooth ground except under special circumstances of projection. What is proved is that, provided the initial spin about the axis exceeds a certain limit, it is possible to assign a limiting value to the inclination of the axis to the vertical which can never at any time be exceeded. This limit depends on the energy, and as the kinetic energy decreases on account of sliding friction

the limiting inclination decreases till, when the energy is reduced to a certain value, the limiting inclination is reduced to zero."

Air resistance and pivot friction, which are excluded from the discussion, finally reduce the spin and make the top fall. Therefore if pure rolling is induced too quickly, as when a darning-egg is spun on a carpet, the body may not have time *to lose enough energy to rise;* on the other hand, if the energy is lost very slowly —body spinning on a plate—the time to rise may be so long as to allow air resistance to reduce the spin below the limiting value.

Gallop assumes the top to rise while its centroid remains at rest. This, however, is not the way in which all tops rise: sometimes while the peg describes a decreasing spiral, the centroid seems to trace a vertical helix of diminishing radius.

60. Least Kinetic Energy. In the case of *steady motion with the centroid at rest,* the least kinetic energy E of the top is

$$E = \tfrac{1}{2}(A\omega_1^2 + B\theta^2 \sin^2 \varphi). \tag{1}$$

Gallop studies the variation of the total energy of the top, kinetic plus potential; we shall find it more convenient to deal only with the kinetic energy.

Equation (1) is conditioned by Jellett's relation, p. 100,

$$A(k + \cos \varphi)\omega_1 + B\theta \sin^2 \varphi = K, \tag{2}$$

and by the assumption of steady motion with no translation. From Ex. 118, p. 101, with $v = 0$,

$$(k + \cos \varphi)\theta = \omega_1. \tag{3}$$

Restriction (3) is necessary because, when pure rolling sets in, no further decrease of kinetic energy can occur for a given inclination φ. For brevity put

$$\cos \varphi = u.$$

Now eliminating ω_1 from (2) and (3), and from (1) and (3), and from the results eliminating $\theta \sin \varphi$, we obtain

$$E = \frac{1}{2}\frac{K^2}{A(k + u)^2 + B(1 - u^2)}, \tag{4}$$

which is the least kinetic energy for steady motion in a given position $u = \cos\varphi$. If the top is to rise beyond this position, the actual kinetic energy must be greater than E, the excess being, however, as small as desired. It must be kept in mind that slip takes place in the actual motion; this is why the actual energy is greater than that in the pure rolling motion.

In Fig. 47 the line whose ordinates represent the kinetic energy for various values of $u = \cos\varphi$ *when there is no friction* must be parallel to the line $P.E.$, representing potential energy, because the distance between them is the constant total energy

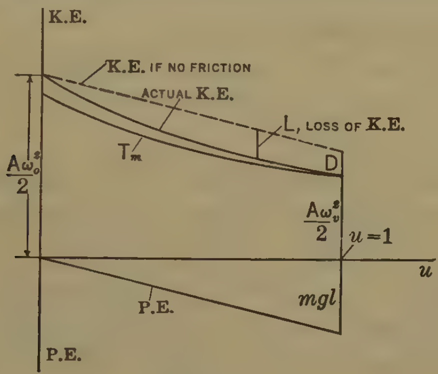


Fig. 47

of the system. Since slipping and consequent dissipation must take place until the vertical position is reached—if the top is to become upright—the loss L must increase until it reaches the greatest value D; otherwise the centroid will not rise higher than some position $u < 1$.

At the beginning of the motion the spin is largest and the circle of contact of the spherical peg is largest; thus the path of the friction force F is greatest and kinetic energy is being converted into heat faster than when the spin gets smaller and the circle of contact gets very small. This of course is subject to the proviso that F does not grow large enough to counteract the decrease of path. But (4), p. 100, shows that F decreases

with diminishing $\dot{\varphi}$, $\dot{\theta}$ and $\sin \varphi$. Or from another point of view, if $\ddot{\eta}$ is the vertical acceleration of the centroid

$$V = m(g + \ddot{\eta}),$$

where $\ddot{\eta}$ is ultimately zero; hence F, which depends on V, also gets smaller.

It follows from these considerations that the *rate* of loss decreases and thus produces an actual kinetic energy curve of the type shown: one therefore that

(i) slopes more than the $P.E.$ line,

(ii) slopes downward,

(iii) is concave upward.

Gallop's criterion for the rise of a top depends on the shape of the energy curve, a point that is sometimes not emphasized enough.

61. Conditions for Rise. As we are finding the least spin necessary for the rise, the actual kinetic energy need never be more than slightly greater than E. For a rise to $u = 1$, the excess must be zero at $u = 1$. In other words, the slope of the E curve *need* never be numerically less than the slope, mgl, of the $P.E.$ line. That is, it may be less, but it is *sufficient* for a slow rise if it is not less. Symbolically, we may write

(1) $$\left|\frac{dE}{du}\right| \geqq mgl,$$

where $| \, |$ denotes the positive numerical value.

Hence from (4), p. 100,

$$\frac{K^2\{A(k+u) - Bu\}}{\{A(k+u)^2 + B(1-u^2)\}^2} \geqq mgl.$$

Therefore, at $u = 1$,

(2) $$K^2 \geqq \frac{mglA^2(k+1)^4}{A(k+1) - B},$$

from which, since K must be real,

(3) $$A(k+1) > B.$$

Equation (3) is satisfied when $A > B$, but only for certain values of k when $A < B$. Another criterion, (7), involving A, B, k will be found presently; both are due to Gallop.

If the top starts at $u_0 = \cos \varphi_0$

$$K = A(k + \cos \varphi_0)\omega_0,$$

and (2) gives

$$\omega_0^2 \geqq \frac{mgl(k+1)^4}{\{A(k+1) - B\}(k + \cos \varphi_0)^2}. \tag{4}$$

Condition (2) satisfies (1) only at $u = 1$. But (1) must be true from $u = 0$ to $u = 1$, and therefore at $u = 0$. This gives

$$K^2 \geqq \frac{mgl(Ak^2 + B)^2}{Ak}.$$

Hence

$$\omega_0^2 \geqq \frac{mgl(Ak^2 + B)^2}{A^3k(k + \cos \varphi_0)^2}. \tag{5}$$

Conditions (4) and (5), also due to Gallop, are independent but the one giving the larger ω_0 must be used. It was shown that the curve of actual kinetic energy is concave upward; since it need lie only a little above E, the E curve is also concave upward. This curvature, greater slope at smaller u, will make (5) true if (4) is true, so that (4) is the proper criterion.

Since the curve slopes downward at a diminishing rate the numerical value of dE/du gets smaller, that is,

$$\frac{d}{du}\frac{dE}{du} < 0. \tag{6}$$

Therefore, using the left member of the equation following (1), we find

$$3(A - B)^2u^2 + 6A(A - B)ku + Ak^2(3A + B) + B(B - A) > 0.$$

This quadratic function, of the form $au^2 + 2bu + c$, since it is never negative, cannot have real roots. This requires the discriminant, $b^2 - 4ac$, to be negative, whence

$$Ak^2 > A - B. \tag{7}$$

This criterion is satisfied by any k provided $A < B$; in this case, (3) must be used. On the contrary, (3) is true for any k when $A > B$, but now (7) sets a lower limit to k.

A top constructed to satisfy (3) and (7) will rise for the velocity specified by (4), but no velocity can make it rise if (3) and (7) are not fulfilled.

Ex. 119. A thin stiff wire is stuck centrally into a ball of radius r, and a circular disk is rigidly mounted on the wire, normal to it, so as to form a top with a spherical peg. If the ball is massless show that the top will not spin upright unless the disk is more than $0.41r$ above the surface of the ball.

Solve the problem when the mass of the ball is not negligible; use numbers instead of letters.

Ex. 120. Prove that (4) requires a larger spin than that in the criterion in § 58.

Ex. 121. Why does a hard-boiled egg spin more easily on its small end?

$A < B$, and k is larger for the spin on the small end: (3), p. 104. The actual rise cannot be followed by the equations: they determine only the ease or stability of the upright spin.

Ex. 122. Explain why (4) demands a larger spin than the condition $A^2\omega^2 > 4mglB$, § 51, for a frictionless top. In the notation of Chapter VI, this condition is $C^2n^2 > 4mglA$.

CHAPTER VIII

CONSTRAINED MOTION

62. Direction of Angular Momentum. Let a solid of revolution, for which $B = C$, spin about its axle at n radians per second while the axle precesses at a rate θ which is not necessarily constant. The axle is one principal axis X, and on account of the symmetry, the axle being the axis of revolution, Y can be taken in the (n, θ) plane, as in Fig. 48, without any loss of generality.

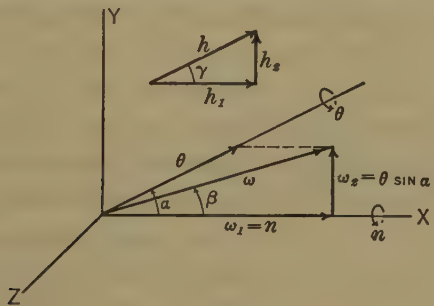


Fig. 48

Since the angular momenta are

$$h_1 = An, \qquad h_2 = C\theta \sin \alpha, \qquad h_3 = 0,$$

the angle γ between X and the resultant momentum h is

$$\tan \gamma = \frac{C\omega_2}{A\omega_1} = \frac{C}{A} \tan \beta = \frac{C}{A} \frac{\theta \sin \alpha}{n}.$$

Hence

$$\gamma \gtreqless \beta \quad \text{if} \quad C \gtreqless A.$$

As C is seldom—although sometimes—greater than A, h generally—but not always—lies near the axle provided n/θ is large. Now from Fig. 48, $\beta > \alpha$ if $n < \theta \cos \alpha$. Hence

$$\gamma > \beta > \alpha \quad \text{when} \quad C > A;$$

thus h lies beyond θ, a case that does not usually occur but that cannot be disregarded.

Since θ determines the velocity of points on the axle, the θ axis passes through the centers of curvature of the loci of all points on the axle.

63. Clinging of the Axle. If a piece of wire bent into any shape, *e.g.*, a spiral, an S, an open triangle, is held so as to touch the physical axle of a gyro spinning on a fixed point, the axle clings to the wire guide—which is to be held at rest—as if it were magnetized. The most astonishing feature of its behavior is the way the axle goes around an end of the wire; in the case of an open triangle, △, the axle follows the configuration around the outside, rounds an end, follows the inside, rounds the other end, and so on.

In order to explain this phenomenon and to see whether it must occur under all circumstances, consider first the simplest case of a gyro supported at its centroid (hollow conical rotor on a needle point) with its axle touching a frictionless guide wire.

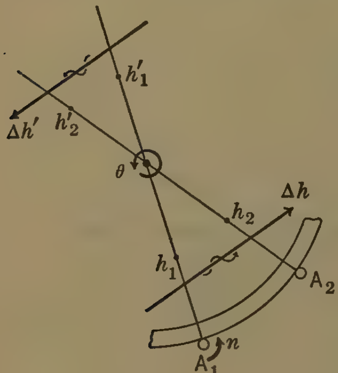


FIG. 49

In Fig. 49, A_1 is the axle as seen from above; n is the spin. The diagram is not a picture of what the eye sees in looking downward at the gyro; it serves merely to indicate the positions of the axle and the θ and h vectors. At the instant illustrated, the axle is precessing about some axis θ lying in the plane containing the axle and the radius of curvature of the guide. The guide defines the cone of constraint (not circular) and may be replaced, at the point of contact, by the intersection of the cone with a plane normal to the axle. The θ axis passes through the center of curvature of this intersection.

As on p. 107, when $A > C$ and n/θ is large, the h axis, h_1 in Fig. 49, lies between A_1 and θ and is near A_1; the next position of h is indicated by its piercing point h_2. The change of angular momentum is the vector difference $h_2 - h_1 = \Delta h$. Since a right-handed moment is required to maintain this change, the guide must press outward in order to exert a right-handed moment about an axis, parallel to Δh, through the point of support. In short, the axle presses against the outside of the wire no matter what the radius of curvature may be.

Now when $C > A$ and n is small, the momentum vector lies beyond θ, and $\Delta h'$ is opposite to Δh. The guide will therefore have to be on the other side of the axle and will not stay in contact in the position shown *unless* θ *is reversed.* Hence if the gyro is given an initial counterclockwise precession the axle will leave the guide.

Friction acts in the sense A_1A_2 on the axle; this makes $h_2 < h_1$; Δh, inclined downward below the plane of the guide, has two components: one forward in this plane and one downward toward the point of support.

That the axle will not always follow the guide may be seen from (4), p. 77:

$$A\ddot{\theta} + (Cn - A\dot{\psi} \cos \theta)\dot{\psi} \sin \theta = M,$$

where M is the moment, about H in Fig. 36, of the guide reaction. Imagine a horizontal elliptic guide wire with its center on the vertical. If the axle moves toward an end of the major axis, $\ddot{\theta}$ is negative because the nutation is retarded. No matter what values M or $\ddot{\theta}$ may have, the solution of this equation as a quadratic in $\dot{\psi}$ shows that it is possible to have

$$A\dot{\psi} \cos \theta > Cn,$$

thus making the parenthesis negative. Hence M will be negative no matter whether $\dot{\psi}$ is $+$ or $-$, and if the axle is on the outer side of the wire (positive M) it cannot remain in contact.

A discontinuity, such as an acute angle in the guide, Fig. 50, still remains to be investigated. Let the axle roll without slipping. The angular momentum vector lies between the guide surface and A, and the reaction on the axle acts toward the right. When the point of contact is at the corner, these conditions still obtain because the corner, if it is geometrically sharp, will be able to offer a reaction in any outward direction necessary to maintain the change of momentum when the axle swings around to the left. Slipping does not alter the case; its only effect is to move the θ axis nearer to A.

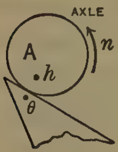


FIG. 50

When the axle is a negligibly thin line the same conditions obtain. A sharp corner has an infinitesimal radius of curvature; this puts the h vector infinitesimally close to the axle. Consequently even if the guide were to vanish suddenly, the axle would continue to turn the corner, since the h vector would remain constant in magnitude and direction.

64. Free Gyro. In the work thus far, we have neglected the rotation of the earth. Its effect on a gyro depends on the nature of the constraint between the gyro and the earth. A free gyro, or one supported at its centroid under no applied forces, maintains the direction of the resultant angular momentum axis unchanged —unchanged, that is, with respect to the so-called fixed stars: absolute fixity of direction has no meaning. The resultant angular velocity vector does not remain fixed in direction; on the contrary, the motion of its end on the invariable plane, § 33, is decidedly complicated. But when the resultant velocity is about the axes of largest or smallest moment of inertia, the spin axis is permanent in the body and in space. (See Chapter V.)

Consider a gyro in latitude φ supported at its centroid by means of frictionless gimbals, Fig. 51. This means of support is often called a *Cardan suspension*, although according to Willis [1]

[1] *Principles of Mechanism*, 1870, p. 439. Willis, p. 438, quotes from a thirteenth-century architect, Wilars de Honecort, who describes a hand-warmer—a charcoal

it dates from the thirteenth century, and Cardan (1501–1575) on seeing one was "unable to assign a use for it." The length of the sidereal day being 86,164 seconds (the mean solar day is 86,400 seconds), the angular velocity of the earth is

$$\Omega = 0.0000729 \textit{ rad. per sec.},$$
$$\Omega^2 = 5.3 \times 10^{-9}.$$

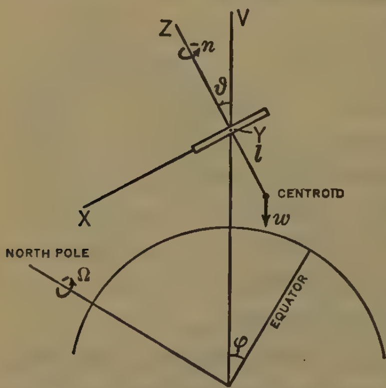
FIG. 51

The components of Ω about the vertical and the tangent to the meridian are $\Omega \sin \varphi$, $\Omega \cos \varphi$, so that in latitude 45° a free gyro would appear to turn once around the vertical and once around the north-south horizontal in about 1.41 days. Friction aside, no spin whatever is necessary in an ideal device like this. For a large spin, or rather a large axial momentum, the slight disturbance due to friction and eccentricity of the pivots will not cause the angular momentum axis to deviate much from the geometric axle; in other words the polhode, § 34, is small.

Shortly after his pendulum experiment in the Pantheon (1851) Foucault tried to show the rotation of the earth by means of a *gyroscope;* the word was coined by him and means, etymologically, *to see rotation.* He suspended a rapidly rotating disk, mounted in gimbals, from a nearly torsionless filament, the axle being horizontal. The apparatus failed to give the expected results principally because he could not keep up the rotation for a long enough time. The same apparatus with an electrically driven rotor—introduced for this purpose by Hopkins in 1878—shows the earth's rotation admirably but does not give satisfactory quantitative values on account of the torsional stiffness of the suspension threads.

brazier supported in *six* gimbal rings to hold it top side up—and adds: ". . . you may turn it about in any way, and the cinders will never fall out. It is excellent for a bishop, for he may boldly assist at high mass, and as long as he holds it in his hands they will be kept warm so long as the fire remains alight."

65. Gyro Axle in the Meridian. The axle of a centroidally supported gyro constrained to remain in the meridian plane sets itself parallel to the earth's axis, or at any rate oscillates about this as an equilibrium position. The reason is obvious: when the axle is parallel to the earth's axis the earth's rotation has no effect on the freely mounted rotor. This idea is also due to Foucault.



V IS VERTICAL, XZ IN THE MERIDIAN, AND Y TOWARD THE WEST

FIG. 52

Consider now the more general case of a gravitationally stable gyro, *i.e.*, a gyro with centroid below the point of support, as in Fig. 52. The ring carrying the axle has trunnions pivoted on a fixed horizontal Y axis that points west. The latitude is φ, the earth's rotation Ω, and the *absolute* spin of the rotor n; n evidently remains constant. The angle between axle and vertical is ϑ.

The velocities of the axes and the rotor are

$$\theta_1 = \Omega \cos(\varphi + \vartheta), \qquad \theta_2 = \dot{\vartheta}, \qquad \theta_3 = \Omega \sin(\varphi + \vartheta),$$

$$\omega_1 = \theta_1, \qquad \omega_2 = \theta_2; \qquad \omega_3 = n \textit{ for rotor}, \qquad \omega_3 = \theta_3 \textit{ for ring.}$$

Let the moments of inertia of the rotor be A, A, C, and of the ring $2C'$, C', C'. Note that the moment of inertia of a ring about a central normal is twice that about a diameter.

The moments on the entire system, rotor and ring, are

L about X: exerted at the trunnions to keep the axle in the plane of the meridian—there is no constraining moment about Z because the rotor is freely mounted on Z;

$\mp M$ about Y: due to friction and *opposite* to $\dot{\vartheta}$;

$-wl \sin \vartheta$ about Y: by w.

The angular momenta are

$$h_1 = (A + 2C')\Omega\cos(\varphi + \vartheta) = I\Omega\cos(\varphi + \vartheta),$$
$$h_2 = (A + C')\dot\vartheta = J\dot\vartheta,$$
$$h_3 = Cn + C'\Omega\sin(\varphi + \vartheta).$$

It is important to understand that the quantities L_0, M_0, N_0, p. 33, are included, indirectly of course, in the weight w which is the resultant of the purely gravitational pull and the centrifugal reaction due to Ω; see § 16.

Ex. 123. Calculate L_0, M_0, N_0 for mass m in Fig. 52.

The results of substituting in (1), p. 32, are

(1) $$L = Cn\dot\vartheta - 2(A + C')\Omega\dot\vartheta\sin(\varphi + \vartheta),$$

$$\mp M - wl\sin\vartheta = (A + C')\ddot\vartheta - Cn\Omega\cos(\varphi + \vartheta)$$

(2) $$+ (A + C')\Omega^2\sin(\varphi + \vartheta)\cos(\varphi + \vartheta),$$

(3) $$0 = (C' - I + J)\Omega\dot\vartheta\cos(\varphi + \vartheta),$$

since $C' - I + J = 0$. The identity (3) verifies the fact that there are no moments about Z. Since $\Omega^2 = 5.3 \times 10^{-9}$, p. 111, the last term in (2) is negligible. Eq. (1) determines the direction of the constraint that keeps the axle in the meridian. Usually n is large and L is then positive; this conforms with Fig. 24 and is moreover in agreement with § 63. For if the trunnions are replaced by a meridianal plane on the west side of the axle, the plane exerts an eastward force on the axle and causes an apparent clinging. The case is reversed when $\dot\vartheta$ is negative. When M, l, and $\ddot\vartheta$ are zero in (2), $\vartheta = 90 - \varphi$ and the axle is parallel to the earth's axis and thus determines the latitude.

66. The Oscillations. With negligible Ω^2 in (2), the equilibrium position ϑ_0 of the axle, found by putting $\ddot\vartheta = 0$, is given by

(4) $$\pm M + wl\sin\vartheta_0 = Cn\Omega\cos(\varphi + \vartheta_0),$$

which, added to (2), gives

$$wl(\sin\vartheta_0 - \sin\vartheta) = (A + C')\ddot\vartheta + Cn\Omega[\cos(\varphi + \vartheta_0) - \cos(\varphi + \vartheta)],$$

whence, with $A + C' = J$,

$$wl\, 2 \sin \frac{\vartheta_0 - \vartheta}{2} \cos \frac{\vartheta_0 + \vartheta}{2} = J\ddot{\vartheta} - Cn\Omega\, 2 \sin \frac{2\varphi + \vartheta_0 + \vartheta}{2} \sin \frac{\vartheta_0 - \vartheta}{2}.$$

Put $\vartheta - \vartheta_0 = \delta$ and assume the motion to be small; then δ is small, but in temperate latitudes ϑ_0 is of the order of magnitude of φ. The last equation becomes

$$(5) \qquad J\ddot{\delta} = -\left[wl \cos \vartheta_0 + Cn\Omega \sin (\varphi + \vartheta_0)\right]\delta.$$

To find the equilibrium position of the axle, plot separately the left and right members of (4) for positive, zero, and negative M, as in Fig. 53. The intersection determines a root of the equa-

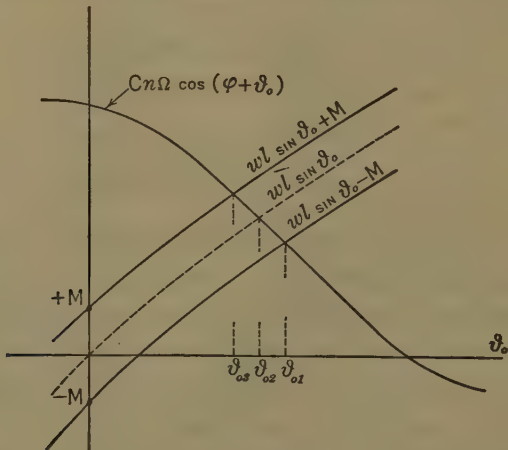


FIG. 53

tion. As the signs in (2) and (4) are opposite, the positive sign $(+ M)$ in (4) corresponds to positive $\dot{\vartheta}$, that is, to motion toward the north pole. Therefore when there is friction, $\vartheta_0 = \vartheta_{03}$ is the equilibrium position for northward motion $+ \dot{\vartheta}$, and $\vartheta_0 = \vartheta_{01}$ is that for southward motion $- \dot{\vartheta}$; $\vartheta_0 = \vartheta_{02}$ is the resting position when there is no friction. If M is small, a slight amount of friction being sufficient to dissipate the kinetic energy and damp the motion, the diagram shows that

$$\vartheta_{01} - \vartheta_{02} = \vartheta_{02} - \vartheta_{03}, \quad \textit{approximately.}$$

This means, as we shall prove, that small damping displaces the equilibrium position through nearly equal angles on the two sides of the equilibrium position of no friction.

Equation (5) is of the simple-harmonic type

$$(7)\qquad \frac{d^2}{dt^2}(\vartheta - \vartheta_0) = -p^2(\vartheta - \vartheta_0), \qquad \vartheta - \vartheta_0 = \delta.$$

Let the axle swing from north to south, from rest at ϑ_1 to rest at ϑ_2; $\vartheta_1 > \vartheta_2$. Multiply (7) by $d\vartheta$, integrate from ϑ_1 to ϑ_2, and note that the velocity $\dot{\vartheta}$ is zero at each limit. There results

$$(8)\qquad \vartheta_1 - \vartheta_0 = \vartheta_0 - \vartheta_2.$$

For a swing from south to north, ϑ_2 to ϑ_3, ϑ_3 being assumed greater than ϑ_0, we find similarly

$$(9)\qquad \vartheta_0 - \vartheta_2 = \vartheta_3 - \vartheta_0.$$

The relations (6), (8) and (9), are pictured in Fig. 54 with exaggerated amplitudes. Since ϑ_{01} and ϑ_{03} are the values of ϑ_0 for south and north motion, ϑ_0 is ϑ_{01} in (8) and ϑ_{03} in (9). From these equations

$$\vartheta_1 - \vartheta_3 = 2(\vartheta_{01} - \vartheta_{03}).$$

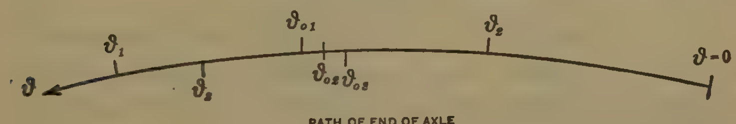


FIG. 54

It follows from Fig. 54 that a north toward south swing is harmonic, (7), about a position at $\vartheta_{01} - \vartheta_{02}$ north of the position ϑ_{02} of no friction; for a northward swing friction shifts the equilibrium position southward by nearly the same amount. Furthermore, in two successive swings farthest north, *e.g.*, ϑ_1, ϑ_3, the axle falls short or loses a displacement $2(\vartheta_{01} - \vartheta_{03})$; similarly for any two consecutive swings toward the south. In other words, amplitudes in the same sense decrease by this amount.

The time of one swing (half cycle) of the periodic motion (5) or (7) is

$$\frac{\pi}{p} = \frac{\pi\sqrt{J}}{\sqrt{wl\cos\vartheta_0 + Cn\Omega\sin(\varphi + \vartheta_0)}} = \frac{\pi\sqrt{J}}{\sqrt{R}},\ \textit{say},$$

and depends on ϑ_0. To see how the time changes with ϑ_0 examine the derivative

$$\begin{aligned}\frac{dR}{d\vartheta} &= -\,wl\sin\vartheta_0 + Cn\Omega\cos(\varphi + \vartheta_0),\\ &= \pm\, M \ (\textit{from}\ (4)).\end{aligned}$$

Note that $M = 0$ for $\vartheta_0 = \vartheta_{02}$. Since R has zero slope at ϑ_{02}, and equal, opposite slopes at the ends of the interval $\vartheta_{01} - \vartheta_{03}$, it is symmetrical in this *small* region and therefore has the same values at ϑ_{01}, ϑ_{03}. Hence the motion is isochronous.

Ex. 124. Draw a rough graph showing the relation between ϑ and time.

67. Axle in Horizontal Plane. When the trunnions of the gimbal ring carrying the rotor are pivoted on a fixed vertical axis Y, Fig. 55, the axle can swing only in a horizontal plane; the supporting structure is omitted.

The velocities of the axes and the body are

$$\theta_1 = -\,\Omega\cos\varphi\sin\vartheta, \quad \theta_2 = \dot{\vartheta} + \Omega\cos\varphi, \quad \theta_3 = \Omega\cos\varphi\cos\vartheta,$$

$$\omega_1 = \theta_1, \quad \omega_2 = \theta_2; \quad \omega_3 = n \textit{ for rotor}, \quad \omega_3 = \theta_3 \textit{ for ring}.$$

With moments of inertia

$$A,\ A,\ C \ \textit{ for rotor}, \qquad 2C',\ C',\ C' \ \textit{ for ring},$$

the angular momenta are

$$\begin{aligned}h_1 &= -\,(A + 2C')\Omega\cos\varphi\sin\vartheta,\\ h_2 &= (A + C')(\dot{\vartheta} + \Omega\sin\varphi),\\ h_3 &= Cn + C'\Omega\cos\varphi\cos\vartheta.\end{aligned}$$

The moments are

L about X: exerted at the trunnions; centroid at origin;

$\mp M$ about Y: by friction and opposite to $\dot{\vartheta}$;

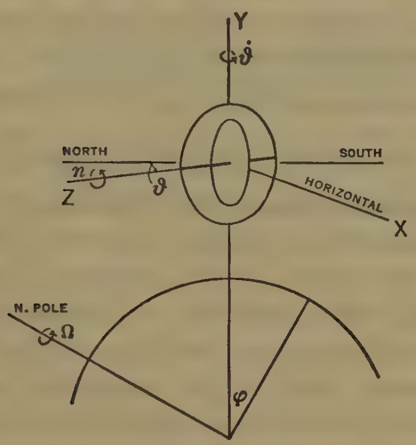
no constraining moment about Z.

The equations of motion, to terms of the first order in Ω and ϑ, are

(1) $$L = Cn\dot{\vartheta},$$

(2) $$\mp M = (A + C')\ddot{\vartheta} + Cn\Omega \cos \varphi \sin \vartheta.$$

Ex. 125. Prove that the Z equation vanishes to terms of first order.



Y IS VERTICAL, XZ HORIZONTAL, AND CENTROID AT THE ORIGIN.

FIG. 55

For small displacements in (2), $\sin \vartheta = \vartheta$; this approximation involves an error of 0.1% for $\vartheta = 4.4°$ and 1.0% for $\vartheta = 14.0°$. The equation is then of the form

(3) $$\ddot{\vartheta} = -k(\vartheta \pm \vartheta_0), \qquad \vartheta_0 = \frac{M}{Cn\Omega \cos \varphi},$$

where the lower sign is to be used for motion from west to east in Fig. 55; and *vice versa.*

The motion is oscillatory, being satisfied by $\vartheta \pm \vartheta_0 = \epsilon \cos \omega t$, and for $\ddot{\vartheta} = 0$ has *two* equilibrium positions $\vartheta = \mp \vartheta_0$, according to whether the swing is from east to west (upper sign) or west to east (lower sign). Let the axle swing eastward (lower sign)

from rest at ϑ_1 to rest at $-\vartheta_2$. Multiplying (3) by $d\vartheta$, integrating, and noting that $\dot\vartheta$ vanishes at each limit, we get

$$\vartheta_1 - \vartheta_2 = 2\vartheta_0.$$

For a swing from ϑ_2 east to ϑ_3 west

$$\vartheta_2 - \vartheta_3 = 2\vartheta_0.$$

Ex. 126. Show the last two relations graphically and prove from the diagram that consecutive amplitudes decrease by $2\vartheta_0$.

Ex. 127. Find the time of a small to and fro swing.

Ex. 128. Prove that the oscillations cannot be small unless M is sufficiently small.

Ex. 129. Can the axle come to rest *within* the position ϑ_0?

68. Tight and Loose Constraints. The axles of the Foucault gyroscopes in Figs. 52 and 55 have one degree of freedom: that in Fig. 52 is free to turn about a horizontal, that in Fig. 55 about a vertical, but in each case the rotation of the earth is made to influence the nutation or the precession by means of a suitable constraint. Each of the constraints may be called *tight* because they constrain the axle to move exactly with the appropriate component of the earth's rotation. A free gyro, supported in gimbals at its centroid—center of the rotor—is under no constraint. If a weight is attached to the axle anywhere except at the center of the rotor, the gyro still has three degrees of *geometric* freedom and the axle has two; each degree represents a possible rotation. Nevertheless the pull of the earth now influences the orientation of the axle and imposes two dynamical constraints (in latitude and longitude circles) without requiring the axle to participate exactly or entirely in the earth's rotation; such constraints may be called *loose*.

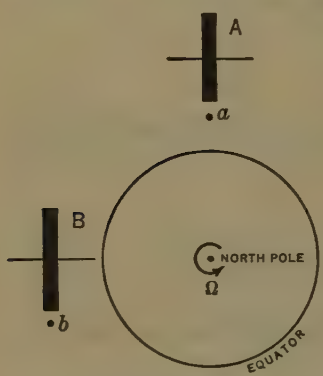


FIG. 56

A type of loose constraint is seen in Fig. 56. At A is shown a rotor mounted with freedom to turn about any axis through its center; the axle is in the equatorial plane and is horizontal. In 6 hours the earth will carry it to the position B without changing the direction of the axle. If a weight is attached below the center, at a, the axle cannot remain in position B, weight at b, even if it can reach it, since the moment of the weight produces rotation about an axis parallel to that of the earth. In this way the pull of the earth would keep the axle horizontal (tangent to the equator) *if the rotor had no spin.* The interaction of spin and the velocity due to the earth's rotation sets up motions to be investigated in Chapter IX.

CHAPTER IX

THE GYRO-COMPASS

69. The Gyro-Pendulum. The device in Fig. 55 constitutes an effective gyroscopic compass, or *gyro-compass*, when installed at a fixed place. It is subject to an error lying between $\pm\ \vartheta_0$ but this can be eliminated by changing the character of the damping to vary with ϑ. Observations by Schuler on the Anschuetz-Kaempfe gyro-compass used at a fixed place in this way, have shown a deviation from the meridian of as little as 20 seconds of arc.[1]

When the gyro is mounted on a ship the tight constraint is replaced by a loose one. The most obvious way of doing this is to make the gyro pendulous by attaching a weight below the point of support. Such a gyro-pendulum represents the basic type of gyro-compass. The axis of the gyro is normally horizontal. If it is normally vertical, the gyro-pendulum then becomes an air-craft instrument for indicating the vertical.

The principle is illustrated diagrammatically in Fig. 57. The method of support, an important practical feature, is not shown, but the system is to be imagined free to turn about the origin of the axes at the center of the rotor, the centroid of the whole body being below the origin.

The north-pointing end of the axle is tilted at ϑ above the horizontal and is swung through an azimuthal angle ψ west of the meridian; φ is the latitude.

The ring carrying the axle is free to swing into any position but we shall here assume it to remain in the vertical plane HV.

The components of the earth's velocity Ω are $\Omega \sin \varphi$ about V

[1] Grammel, *Die mechanischen Beweise für die Bewegung der Erde*, 1922, p. 66. For a detailed account of the gyro-compass see Rawlings, *The Gyroscopic Compass and its Deviations*, 1929.

and $\Omega \cos \varphi$ about N; these are resolved into

$$
\begin{array}{llll}
\Omega \sin \varphi \cos \vartheta & \textit{about} & Y, & \\
\Omega \sin \varphi \sin \vartheta & \text{“} & X, & \\
\Omega \cos \varphi \cos \psi & \text{“} & H \left\{ \begin{array}{lll} \Omega \cos \varphi \cos \psi \cos \vartheta & \textit{about} & X, \\ -\ \Omega \cos \varphi \cos \psi \sin \vartheta & \text{“} & Y, \end{array} \right. \\
\Omega \cos \varphi \sin \psi & \text{“} & Z. &
\end{array}
$$

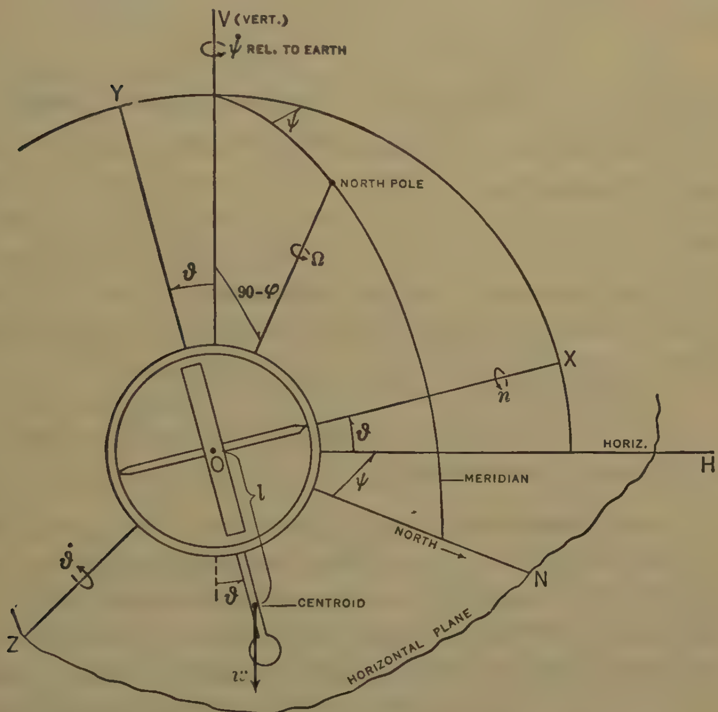


Fig. 57

There are also the velocities of the frame

$$\dot{\psi} \sin \vartheta \quad \textit{about } X, \qquad \dot{\psi} \cos \vartheta \quad \textit{about } Y, \qquad \dot{\vartheta} \quad \textit{about } Z.$$

Hence

$$\dot{\theta}_1 = \Omega \sin \varphi \sin \vartheta + \Omega \cos \varphi \cos \psi \cos \vartheta + \dot{\psi} \sin \vartheta,$$

$$\dot{\theta}_2 = \Omega \sin \varphi \cos \vartheta - \Omega \cos \varphi \cos \psi \sin \vartheta + \dot{\psi} \cos \vartheta,$$

$$\dot{\theta}_3 = \Omega \cos \varphi \sin \psi + \dot{\vartheta},$$

and

$$\omega_1 = n \quad \textit{for rotor}, \qquad \omega_1 = \theta_1 \quad \textit{for frame;} \qquad \omega_2 = \theta_2, \qquad \omega_3 = \theta_3,$$

where n is absolute. As n is large, 6,000 to 20,000 r.p.m., the difference between absolute and relative is insignificant.

If the moments of inertia are

$$A,\ C,\ C \quad \textit{for rotor}, \qquad A',\ B',\ C' \quad \textit{for frame},$$

and if

$$B' + C = I, \qquad C + C' = J,$$

$$h_1 = An + A'(\Omega \sin \varphi \sin \vartheta + \Omega \cos \varphi \cos \psi \cos \vartheta + \dot{\psi} \sin \vartheta),$$

$$h_2 = I(\Omega \sin \varphi \cos \vartheta - \Omega \cos \varphi \cos \psi \sin \vartheta + \dot{\psi} \cos \vartheta),$$

$$h_3 = J(\Omega \cos \varphi \sin \psi + \dot{\vartheta});$$

hence

$$\dot{h}_1 = A'(\Omega\dot{\vartheta} \sin \varphi \cos \vartheta - \Omega\dot{\psi} \cos \varphi \sin \psi \cos \vartheta$$
$$- \Omega\dot{\vartheta} \cos \varphi \cos \psi \sin \vartheta + \ddot{\psi} \sin \vartheta + \dot{\psi}\dot{\vartheta} \cos \vartheta),$$

$$\dot{h}_2 = I(- \Omega\dot{\vartheta} \sin \varphi \sin \vartheta + \Omega\dot{\psi} \cos \varphi \sin \psi \sin \vartheta$$
$$- \Omega\dot{\vartheta} \cos \varphi \cos \psi \cos \vartheta + \ddot{\psi} \cos \vartheta - \dot{\psi}\dot{\vartheta} \sin \vartheta),$$

$$\dot{h}_3 = J(\Omega\dot{\psi} \cos \varphi \cos \psi + \ddot{\vartheta}).$$

70. Small Quantities. For the purpose of investigating the small motions of the gyro-pendulum—the large motions require extraordinarily complicated analysis—it is sufficient to retain only small quantities of the first order. For instance, $\Omega^2 = 5.3 \times 10^{-9}$ is certainly negligible and so are such products as $\dot{\psi}\dot{\vartheta}$, $\ddot{\psi}\vartheta$, when ψ, ϑ are small. Although $\Omega\dot{\vartheta}$, Ω^2, $\dot{\vartheta}^2$ are of the same order of magnitude, their effect in the equations of motion is different. Rejecting Ω^2 makes a *numerical* difference and is equivalent to ignoring a small change of variable or transformation of co-ordinates; rejecting $\Omega\dot{\vartheta}$ changes the character of the motion radically.

The effect may be illustrated by a simple example. Let

$$\dot{v} = 1 + v + v^2,$$

where, say, $v = 0$ when $t = 0$. The v, t relation can be found by integration but the general form of its graph, which is sufficient for the illustration, can be plotted directly from the slope. The effect of omitting v^2 in one case, and both v and v^2 in the other, is seen in Fig. 58.

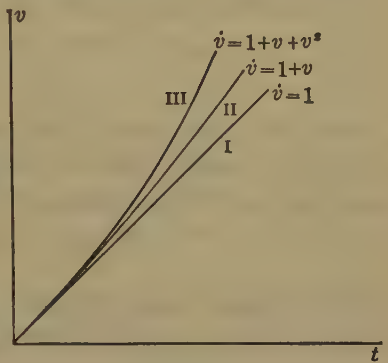


FIG. 58

For small t the velocity is small since it has not had time enough to suffer much change. All three curves give about the same v for, say, $t = 0.01$ sec. However, curve I represents the acceleration $\dot{v}$ as constant whereas both II, III show it to increase with time, the increase in these two cases being of nearly the same character and differing only slightly in magnitude. In short, curve II is much more faithful than I and about as good as III.

Consider now the difference between dropping a constant, say Ω^2, and a variable, say $\Omega\vartheta$. Take the case of

$$\ddot{x} = a - x.$$

The motion is harmonic whether a is large or small, so that omitting a has no effect on the *type* of the motion. Rejecting x if it is small completely hides its oscillatory nature, and while the numerical results may be close enough, they often lead to unjustifiable inferences.

71. The Equations of Motion. It has just been shown that it is not safe to omit a term of the type $\Omega\vartheta$ even when Ω^2, ϑ^2 are negligible. The moments on the gyro-pendulum, Fig. 57, are, in the notation of § 20,

$L = 0$, $M = 0$, since there are no constraints about X, Y;
$N = -wl\vartheta$; $L_0 = M_0 = N_0 = 0$ as in § 65.

Hence for small motions, with $\sin\vartheta = \vartheta$ and $\sin\psi = \psi$, (1), p. 32, give

$$0 = (A' - I + J)\Omega\dot{\vartheta}\sin\varphi,$$
$$0 = I\ddot{\psi} + [An + \Omega(A' - I - J)\cos\varphi]\dot{\vartheta} + An\Omega\psi\cos\varphi,$$
$$-wl\vartheta = J\ddot{\vartheta} - [An + \Omega(A' - I - J)\cos\varphi]\dot{\psi} + An\Omega\vartheta\cos\varphi - An\Omega\sin\varphi.$$

Since $\Omega\dot{\vartheta}$ is not negligible, the first equation shows that the centroid will not stay in the vertical plane through the point of support, that is, $L \neq 0$, unless, as $\dot{\vartheta} \neq 0$, $A' - I + J = 0$ or, p. 122,

(1) $$B' = A' + C'.$$

This relation can be satisfied only by a lamina (a flat plate) in the XZ plane; the frame then should lie as nearly as possible in this plane. While this condition is practically unrealizable, it shows at any rate that the horizontal ring used in gyrocompasses should be much more massive than the parts above or below the XZ plane, and also that a small l with a large w is better than the reverse.

It is evident from the absence of products of inertia in the equations of motion that XYZ are principal axes. The motion of the point of support when the compass is mounted on a ship produces oscillations about horizontal axes through that point. In order that the system may be in balance about these axes, Ex. 73, p. 45, they must be principal. To satisfy this condition as well as (1), the mass should be distributed in a circle about Y.[2]

PROPERTIES OF ROTOR[1]

Compass	Weight lbs.	Rad. of gyr. inches	Ang. vel. r.p.m.	An lb.-ft.^{2}
Anschuetz	6.	1.85	20,000	9.
Brown..............	4.5	1.57	15,000	3.6
Sperry, mark V	50.	4.62	8,600	210.

[1] Data from *Encyc. Brit.*, vol. 30, p. 733 (1922).
[2] See also Rawlings, *The Gyroscopic Compass and its Deviations*, Chap. VII.

The Ω term in the brackets on p. 124 is negligible compared with An, values of which are given in the table.
With this approximation and the abbreviations

(2) $N = An, \qquad a = An\Omega \cos \varphi, \qquad b = An\Omega \cos \varphi + wl,$

the equations of motion are

(3) $$I\ddot{\psi} + N\dot{\vartheta} + a\psi = 0,$$

(4) $$J\ddot{\vartheta} - N\dot{\psi} + b\vartheta - N\Omega \sin \varphi = 0.$$

In the Sperry compass, Mark V, $wl = 6.3$ ft.-lbs., p. 143, and $An\Omega \cos \varphi = 0.014$ ft.-lbs. at $\varphi = 0$, which is less than a quarter per cent of wl.

72. Gravity Moment and Directive Moment. The tilt ϑ is opposed by the *gravity moment* $wl\vartheta$. The gravity moment, in acting to level the axle, produces a slow precession $\dot{\psi}$ about the vertical. When the effect of Ω is disregarded the greatest steady value of $\dot{\psi}$ is given by

$$wl\vartheta = An\dot{\psi},$$

a special case of (4). For the position in Fig. 57, $\dot{\psi}$ turns the axle away from the meridian unless n and $wl\vartheta$ have opposite signs to make $\dot{\psi}$ negative. If $wl\vartheta$ is to precess the axle northward, ϑ should change sign with ψ and vanish at $\psi = 0$. The requisite connection or coupling between ϑ and ψ is obtained by the loose gravitational constraint discussed in § 68, and stated in (3) and (4). The result of the coupling is a vertical *directive moment* assisting the gravity moment in directing the axle to the north-south position. To understand its action compare

(i) $$I\ddot{\psi} + a\psi = 0,$$

obtained by putting $\dot{\vartheta} = 0$ in (3), with

(ii) $$I\ddot{\alpha} + wl\alpha = 0,$$

the equation of motion of a compound pendulum swung through a small angle α.

The restoring or righting moment on the pendulum is $wl\alpha$, and the corresponding term $a = An\Omega\psi \cos\varphi$ in (i) is the restoring or directive moment on the axle. The factor ψ was derived from $\sin\psi$; consequently the directive moment for a large azimuth is $An\Omega \cos\varphi \sin\psi$.

The largest gravity and directive moments are those of the Sperry compass, Mark V; for example $wl\vartheta = 0.33$ ft.-lb. at $\vartheta = 3°$ in latitude 30°–50°, and $An\Omega \cos\varphi \sin\psi$ is 0.014 ft.-lb. at $\varphi = 0$, $\psi = 90°$, but only 0.00017 ft.-lb. at $\varphi = 45°$, $\psi = 1°$. It is evident in view of these small values that any appreciable amount of friction about the vertical axis destroys the precision of a gyro-compass.

The largest compass is the new Sperry, Mark X; its data are given in the following table.

Mark X Compass Data

Weight of Gyro	120 lbs.
" " Gyro and Case	180 "
" " Compass complete in Binnacle	1350 "
Moment of Inertia of Rotor	22.2 lbs. ft.^{2}
Lever arm of ballistic containers	7.25 in.
" " " " connection arm	7.825 in.
Eccentricity of Damping Connection	.230 in.
wl in Lat. of Brooklyn Navy Yard	20.2 lbs. ft.
Diameter of Gyro	13.625 in.
Width of Gyro	3.5 in.
Speed of Gyro	10,000 R.P.M.
A.-C. voltage of supply to gyro	105
A.-C. amperes of supply to gyro	2.2
Watts required to run gyro	225
Temperature rise of gyro case	40° C.
Undamped period in Lat. of Brooklyn, N. Y.	84.3′
Damped period in Lat. of Brooklyn, N. Y.	89.0′
Percentage of damping	67%
Power Factor of Gyro A.-C. Supply	56.3%
Tilt of gyro	5.85′
Maximum diameter of compass binnacle	50.5″
Approximate depth of compass binnacle and cover	53″
Maximum diameter of cardan ring	40¼″
Maximum dimensions of compass spider	29″ x 34$\frac{3}{16}$″
Maximum diameter of base of binnacle	39″
Maximum diameter of base bolt holes	35″
Maximum diameter of binnacle cover	51″
Damping Angle	1° 41′
Frequency of A.-C. Supply to Gyro	350

73. Stability. If the axle is brought to rest in the meridian it will remain inclined at some angle ϑ_ϵ, the resting tilt, to the horizontal. From (4), with $\ddot{\vartheta} = 0$ and $\dot{\psi} = 0$,

(5) $$\vartheta_\epsilon = \frac{An\Omega \sin \varphi}{An\Omega \cos \varphi + wl},$$

which is about 6 minutes of arc for the Sperry V at $\varphi = 45°$; the gravity moment in this case is 0.018 ft.-lb.

Equation (5) can be got from first principles, for when the axle turns with the meridian ($\psi = 0$), $-wl\vartheta_\epsilon$ maintains the vector change-rate about Z of the X, Y components of angular momentum. The axle being nearly horizontal in the meridian, the angular momentum of the rotor about N, Fig. 57, is nearly An and turns about V at $\Omega \sin \varphi$, thus producing $-An\Omega \sin \varphi$ about Z. Also $\Omega \cos \varphi$, the velocity of the frame about H, has a component $\vartheta_\epsilon \Omega \cos \varphi$ about $-Y$ and turns An toward Z. Hence

$$-wl\vartheta_\epsilon = -An\Omega \sin \varphi + An\Omega\vartheta_\epsilon \cos \varphi.$$

When $An\Omega \cos \varphi$ is negligible in (5)

$$wl\vartheta_\epsilon = An\Omega \sin \varphi,$$

which means that $wl\vartheta_\epsilon$ precesses the axle about the vertical at the earth's rate $\Omega \sin \varphi$; this is necessary, of course, if the axle is to stay in, and turn with, the meridian.

We shall now use the kinetic energy to test the stability of the rotor in the equilibrium position ϑ_ϵ. First however let us make (3) and (4) symmetric by changing the variable from ϑ to

$$\nu = \vartheta - \vartheta_\epsilon,$$

whence

(6) $$I\ddot{\psi} + N\dot{\nu} + a\psi = 0,$$

(7) $$J\ddot{\nu} - N\dot{\psi} + b\nu = 0.$$

Multiplying (6) by $d\psi$, (7) by $d\nu$, adding, and integrating, we get

(8) $$E = \tfrac{1}{2}(I\dot{\psi}^2 + J\dot{\nu}^2) = k - \tfrac{1}{2}(a\psi^2 + b\nu^2),$$

where E is all the kinetic energy except that of the rotor about its axle, and k is the constant of integration. $\frac{1}{2}(a\psi^2 + b\nu^2)$ corresponds to potential energy and is called the *force function* because the forces (or moments) are the ψ and ν derivatives of it. Coordinates like ψ, ν, which with their derivatives do not occur as *products* in the kinetic energy and the force function, are called *principal* and are useful in the general study of oscillating systems.

A pendulum or a vibrating elastic body has maximum kinetic energy in its equilibrium position; for the maximum energy of the gyro-pendulum, we find

$$\frac{dE}{dt} = - a\psi\dot{\psi} - b\nu\dot{\nu} = 0,$$

which is satisfied by $\psi = 0$ and $\nu = 0 = \vartheta - \vartheta_\epsilon$. This is the equilibrium position provided E is a maximum, *i.e.*, provided d^2E/dt^2 is negative, which is true. From this we may infer that the gyro axle oscillates about the position $\psi = 0$, $\vartheta = \vartheta_\epsilon$. If it does, the solution of (6) and (7) is likely to be of the form

$$\psi = \psi_0 \sin (pt + \epsilon), \qquad \nu = \nu_0 \cos (pt + \epsilon), \tag{9}$$

which is arrived at by *trial*, a not unusual way of solving differential equations. A formal method is given later.

By substitution it is seen that (9) is a solution of (6) and (7) for suitable values of ψ_0, ν_0, p. Hence the axle has two simple harmonic motions at 90° to each other; their periods are equal, and their phases differ by 90°, since $\nu = \nu_0$ when $\psi = 0$. The end of the axle describes an ellipse of semi-axes proportional to ψ_0, ν_0. Let it start in the meridian, $\psi = 0$, with $\dot{\nu} = 0$ but $\dot{\psi} \neq 0$. Then at $t = 0$

$$\psi = \psi_0 \sin \epsilon = 0,$$
$$\dot{\nu} = - p\nu_0 \sin \epsilon = 0,$$

each of which gives $\epsilon = 0$. Hence from (9)

$$\psi = \psi_0 \sin pt, \qquad \nu = \nu_0 \cos pt. \tag{10}$$

74. The Periods. From (6), (7), (10)

$$(11)\qquad -I\psi_0 p^2 - N\nu_0 p + a\psi_0 = 0,$$

$$(12)\qquad -J\nu_0 p^2 - N\psi_0 p + b\nu_0 = 0.$$

Hence

$$(13)\qquad \frac{\nu_0}{\psi_0} = \frac{a - Ip^2}{Np} = \frac{Np}{b - Jp^2}$$

or

$$(14)\qquad IJp^4 - (N^2 + bI + aJ)p^2 + ab = 0,$$

whence, if N^2 is much larger than $bI + aJ$,

$$p^2 = \frac{N^2}{2IJ}\left(1 \pm \sqrt{1 - \frac{4abIJ}{N^4}}\right).$$

Since N is large compared with a, b, I or J, we need retain only two terms in the expansion of the radical into its power series; the approximate—but numerically correct—value of p^2 is

$$p^2 = \frac{N^2}{2IJ}\left[1 \pm \left(1 - \frac{2abIJ}{N^4}\right)\right].$$

It is seen from the values of a, b, N on p. 125 that ab/N^2 is extremely small; consequently there are two decidedly unequal angular frequencies p:

$$(15)\qquad p_1^2 = \frac{N^2}{IJ}, \qquad p_2^2 = \frac{ab}{N^2}.$$

The corresponding periodic times are

$$(16)\qquad T_1 = 2\pi\sqrt{\frac{IJ}{N^2}},\quad T_2 = 2\pi\sqrt{\frac{N^2}{ab}} = 2\pi\sqrt{\frac{N}{wl\Omega\cos\varphi}},$$

where T_1 represents a fast tremor, and T_2 a slow oscillation several hundred times as long. At $\varphi = 45°$ and for $wl = 6.3$ ft.-lbs. and $N = 210$, the slow period is 85 minutes.

The ratio of the amplitudes, (13), is astonishing: for $\varphi = 45°$, $wl = 6.3$, and $N = 210$, the precession has nearly 18 times the amplitude of the nutation. This means that (6), (7), which

are true only for small swings, involve an appreciable error unless ν_0 is considerably less than one degree.

There is a curious relation between T_1 and T_2, and the periods obtained by neglecting the velocity terms in (6), (7). Thus if ν were constant in (6) the period of the ψ motion would be $T_3 = 2\pi\sqrt{I/a}$; if ψ were constant in (7), that of ν would be $T_4 = 2\pi\sqrt{J/b}$, whence

$$T_1 T_2 = T_3 T_4.$$

If the fast period, which merely imparts a tremulous character to the elongated ellipse, is neglected, the solution is independent of I and J; this suggests that the acceleration terms in (6) and (7) can be dropped without serious error. There results

$$N\dot{\nu} + a\psi = 0,$$
$$-N\dot{\psi} + b\nu = 0,$$

whence, after differentiation and elimination,

$$N^2\ddot{\nu} + ab\nu = 0, \tag{17}$$

$$N^2\ddot{\psi} + ab\psi = 0, \tag{18}$$

which represent simple harmonic motion of period T_2, (16), and are satisfied by the ratio of amplitudes, (13), when p^2 is negligible; they are therefore equivalent to (6) and (7).

It was shown that the Ω term in (2), p. 125, is generally a negligible part of wl, and for all practical purposes $b = wl$. If $l = 0$, the equations fail completely to give a correct representation of the motion: for example, (16), (17) and (18), with $b = 0$, are wrong, numerically as well as in principle. This illustrates the care necessary in using approximations. It is necessary to use $b = An\Omega \cos \varphi$ when $l = 0$; from (16)

$$T_2 = \frac{2\pi}{\Omega \cos \varphi},$$

and from (5),

$$\vartheta_\epsilon = \tan \varphi.$$

When there is no gravitational constraint the axle preserves its direction in space and stays inclined at an angle ϑ_ϵ (really at an angle arctan $\vartheta_\epsilon = \varphi$) above the horizontal; that is, it remains parallel to the earth's axis.

75. Damping. The gyro-pendulum does not function properly as a gyro-compass unless means are used to suppress or damp its oscillations and make it settle as nearly as possible in the meridian. We shall suppose this to be accomplished by a resisting or damping moment $-\mu\dot{\vartheta}$ applied about the Y axis of Fig. 57. The Sperry and the old type Anschuetz are damped in this way, μ being 0.11 to 0.35 ft.-lb. in the former and about 0.035 in the latter. In the Arma [1] and the Brown,[2] the damping moment is applied about the Z axis (see § 92). Both methods are effective since the ψ and ν motions are coupled to be interdependent. The axle of a damped gyro-compass settles to within 1° of the meridian in three hours or so; the magnetic needle takes less than as many minutes.

When $-\mu\dot{\vartheta}$ is put into the left member of the Y equation on p. 124, the approximate equations on p. 130 become

(1) $$\mu\vartheta + N\dot{\nu} + a\psi = 0,$$

(2) $$-N\dot{\psi} + b\nu = 0.$$

The resting position, for which $\dot{\psi} = \dot{\nu} = 0$, is

$$\nu = 0 \quad \text{or} \quad \vartheta = \vartheta_\epsilon \quad [\textit{see } (5), \text{ p. } 127]$$

and

$$\mu\vartheta_\epsilon + a\psi_\epsilon = 0,$$

whence from (2), p. 125,

(3) $$\psi_\epsilon = -\frac{\mu \tan \varphi}{An\Omega \cos \varphi + wl}.$$

Therefore the axle of a vertically damped compass comes to rest at ψ_ϵ *east* of the meridian and does not point true north except at

[1] A modification of the Anschuetz; made by the Arma Engineering Company, New York.

[2] *Engineering*, vol. 109, Feb. 13, 1920, p. 202.

the equator. For this reason ψ_e is called the latitude or *damping error*. The error can be eliminated by putting a small weight on the *north* end of a right-handed spinning compass sufficient to make $\vartheta = 0$. Equations (1) and (2) then give for the resting position, if the moment of the weight is called m,

$$\mu\vartheta + a\psi_e = 0, \qquad m + b(\vartheta - \vartheta_e) = 0,$$

from which, for $\psi = 0$ and $\vartheta = 0$, *i.e.*, if the axle is *perfectly horizontal*,

$$m = b\vartheta_e = An\Omega \sin \varphi. \tag{4}$$

In this position the gravity and damping moments vanish and the correcting moment m produces just the right precession to enable the axle to keep up with the meridian plane. Instead of correcting the error, it may be compensated for by turning the lubber line—a line parallel to the ship's keel—toward the east so that the angle between true north and keel equals that between compass north and the lubber line.

Equations (1) and (2) become more symmetric if ψ is changed to a new variable

$$\gamma = \psi - \psi_e, \tag{5}$$

and if ϑ is replaced by $\nu + \vartheta_e$. Using the relation preceding (3), we get

$$\mu\nu + N\dot{\nu} + a\gamma = 0, \tag{6}$$

$$-N\dot{\gamma} + b\nu = 0. \tag{7}$$

76. Stability. It remains to be proved that the damping moment brings the axle to rest. As in § 73, multiply (6), above, by $d\gamma$, (7) by $d\nu$, add, and integrate from state 1 to state 2. There results

$$\int_1^2 \mu\nu d\gamma = P_1 - P_2, \tag{1}$$

where

$$P = \tfrac{1}{2}(a\gamma^2 + b\nu^2)$$

is the potential energy at any instant. P is like the potential

energy $cs^2/2$ of a stretched spring, s being the stretch and c the force per unit stretch. The integral, the work done by the damping, is always positive since $\mu\nu$ reverses direction with $d\gamma$. As it gets larger, P_2 gets smaller and γ, ν approach zero. The kinetic energy is absent from (1) because that part of the system to which I, J refer is assumed to be nearly massless.

When the inertia of the system is not neglected the stability cannot be determined from energy considerations. The obvious alternative is to examine the solution of the equations of motion. With the acceleration turns retained, (6) and (7), p. 132 (compare (6), (7), p. 127), are

$$I\ddot{\gamma} + N\dot{\nu} + a\gamma + \mu\nu = 0, \tag{2}$$

$$J\ddot{\nu} - N\dot{\gamma} + b\nu = 0. \tag{3}$$

The derivative of (2) contains $\dot{\gamma}$, $\dddot{\gamma}$, which can be replaced by their values from (3); there results

$$IJ\ddddot{\nu} + (N^2 + aJ + bI)\ddot{\nu} + \mu N\dot{\nu} + ab\nu = 0. \tag{4}$$

Since (4) is satisfied by

$$\nu = e^{\lambda t},$$

we have

$$IJ\lambda^4 + (N^2 + aJ + bI)\lambda^2 + \mu N\lambda + ab = 0. \tag{5}$$

Equation (5) does not reduce to (14), p. 129, when $\mu = 0$, because λ and p *have different meanings.*

If λ is real and positive, or if it is complex, say $\lambda = x + iy$, with its real part x positive, ν grows indefinitely and the motion is unstable.

Descartes' rule of signs shows that there are no real positive roots, but possibly two real negative roots. The four roots are thus

$$-\lambda_1 \pm i\lambda_2, \quad +\lambda_3 \pm i\lambda_4,$$

where λ_2 might be zero for certain values of the coefficients in (5). Since there is no λ^3 term, we have

$$-\lambda_1 + \lambda_3 = 0.$$

When the IJ term is negligible, the two roots of the remaining quadratic are given by (4), p. 135; these are approximately roots of (5) above. Hence $\lambda_2 \neq 0$, and all four are complex.

The instability of the motion (λ_3 being positive) may be attributed to the damping or to the *exact* equality of λ_1 and λ_3. But it is conceivable that if friction were not neglected, (5) would contain a small λ^3 term that would permit λ_3 to be negative; this is the case in the Sperry—see § 89. Or, from another point of view: the motion defined by (5) has nearly the same periods as (15), p. 129, the slow motion being truly damped by μ and the fast tremor being extinguished by the *neglected* friction.

It is curious that the method of damping which is successful in practice is theoretically unsound (friction being *neglected*) when I, J are too large to be rejected. But the way the damping device functions when I, J are small is still more curious. In this case the motion is defined by (1) and (2), p. 135, which can be written in the form

$$\frac{N^2}{wl}\ddot{\gamma} + \frac{\mu N}{wl}\dot{\gamma} + N\Omega\gamma\cos\varphi = 0, \tag{6}$$

$$\frac{N}{\Omega\cos\varphi}\ddot{\nu} + \frac{\mu}{\Omega\cos\varphi}\dot{\nu} + wl\nu = 0. \tag{7}$$

These resemble the equation of motion of a pendulum swinging in a resisting medium:

$$I\ddot{\theta} + k\dot{\theta} + wl\theta = 0. \tag{8}$$

The dimensions of N^2/wl and $N/\Omega\cos\varphi$ are ML^2 of moment of inertia and correspond to I; since they are very large, the system behaves as if it had great inertia. Like k, $\mu N/wl$ and $\mu/\Omega\cos\varphi$ function as damping coefficients: observe that both the γ and ν motions are apparently damped by resistances varying with the velocity. $N\Omega\cos\varphi$ and wl, of dimensions ML^2T^{-2}, are the restoring or righting moments per radian. In view of the magnitudes of these quantities (large inertia and resistance, small moment) it will not be surprising to find that the period

is long and that the motion is quickly extinguished; the rejection of the I, J terms introduces no sensible error.

77. Small Damped Oscillations. The result of eliminating ν, $\dot{\nu}$ from (6), p. 132, by means of (7), and similarly $\dot{\gamma}$ from the derivative of (6), is

$$\ddot{\gamma} + \frac{\mu}{N}\dot{\gamma} + p^2\gamma = 0, \tag{1}$$

$$\ddot{\nu} + \frac{\mu}{N}\dot{\nu} + p^2\nu = 0, \tag{2}$$

where

$$p^2 = \frac{ab}{N^2}. \tag{3}$$

Equation (2) is satisfied by $\nu = e^{\lambda t}$; substitution gives

$$\lambda^2 + \frac{\mu}{N}\lambda + p^2 = 0,$$

the roots of which are

$$\lambda_{1,2} = -\frac{\mu}{2N} \pm \sqrt{\frac{\mu^2}{4N^2} - p^2}. \tag{4}$$

Since (2) is of the second order, its solution must contain two arbitrary constants; hence the complete solution is

$$\nu = k_1e^{\lambda_1 t} + k_2e^{\lambda_2 t}. \tag{5}$$

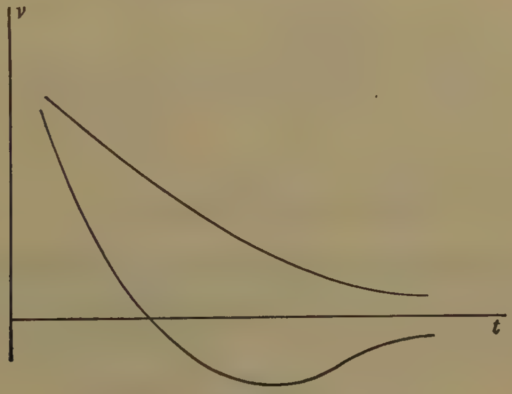


FIG. 59

When μ is sufficiently large, for example, 0.50 ft.-lb. in the Sperry, $p^2 < \mu^2/4N^2$ and λ is real and negative. Hence ν keeps on decreasing with increase of time and does not become zero until $t = \infty$. It may however pass through zero once in a finite period of time if k_1 and k_2 have opposite signs. This motion is called *dead-beat* or *aperiodic*, both cases of it being illustrated in Fig. 59.

If $p^2 > \mu^2/4N^2$, λ is complex and we may write

$$(6) \qquad \lambda_{1,2} = -\frac{\mu}{2N} \pm iq, \qquad \text{where} \qquad q = \sqrt{p^2 - \frac{\mu^2}{4N^2}}.$$

By De Moivre's theorem, we have

$$e^{x \pm iy} = e^x \cos y \pm ie^x \sin y.$$

Hence

$$e^{\lambda t} = e^{-(\mu t/2N)}(\cos qt \pm i \sin qt)$$

and (5) can be put into the form

$$(7) \qquad \nu = e^{-(\mu t/2N)}(c_1 \cos qt + ic_2 \sin qt).$$

Similarly

$$(8) \qquad \gamma = e^{-(\mu t/2N)}(c_3 \cos qt + ic_4 \sin qt).$$

To find c_1, c_2, c_3, c_4, let the time be measured from the instant the axle passes through its resting position $\gamma = 0$; then

$$(9) \qquad 0 = c_3, \quad \text{and} \quad \gamma = e^{-(\mu t/2N)}(ic_4 \sin qt).$$

If $\nu = \nu_0$ when $t = 0$, (7) gives $c_1 = \nu_0$.

Since $N\dot{\gamma} = b\nu$, p. 132, for any t and therefore for $t = 0$,

$$Nic_4q = bc_1 = b\nu_0 \qquad \text{or} \qquad Nqic_4 = b\nu_0.$$

Hence (9) becomes

$$(10) \qquad Nq\gamma = b\nu_0 e^{-(\mu t/2N)} \sin qt.$$

For the *same phase* of γ, $\gamma = 0$ for $t = 0$ and for every time-increase of $T = 2\pi/q$; T is the period of the damped motion. The ratio of any two successive γ's, in the same phase, at times t, $t + T$, is

$$\gamma_2/\gamma_1 = e^{-(\mu T/2N)} = e^{-(\pi\mu/Nq)}.$$

It is easy to verify that successive maxima ($\dot{\gamma} = 0$) in the same phase also follow at intervals T. The exponent $\mu T/2N$ is called the *logarithmic decrement.*

The value of ν is found by substituting from (7), (10) into $N\dot{\gamma} = b\nu$; this gives

$$-\frac{\nu_0 \mu}{2Nq} = ic_2,$$

whence

$$\nu = \frac{\nu_0}{q} e^{-(\mu t/2N)} \left(q \cos qt - \frac{\mu}{2N} \sin qt \right).$$

But from (6),

$$q^2 + \frac{\mu^2}{4N^2} = p^2.$$

Therefore, if

(11) $$\frac{q}{p} = \cos \epsilon, \qquad \frac{\mu}{2Np} = \sin \epsilon,$$

(12) $$\nu = \frac{p\nu_0}{q} e^{-(\mu t/2N)} \cos (qt + \epsilon).$$

Equations (10) and (12) represent damped oscillations at right angles to each other; they have the same period, their amplitudes decrease in the same ratio, and their phase difference is ϵ. From p. 129, the period of the undamped motion is

(13) $$\frac{2\pi}{p} = 2\pi \sqrt{\frac{N}{wl\Omega \cos \varphi}};$$

that of the damped oscillations is longer; by (6) its value is

(14) $$T = \frac{2\pi}{q} = \frac{2\pi}{\sqrt{\dfrac{wl\Omega \cos \varphi}{N} - \dfrac{\mu^2}{4N^2}}}.$$

78. Locus of the Axle. For the reason explained in § 86 the free period of all gyro-compasses is made 85 minutes, that is, by (13),

$$85 \times 60 = \frac{2\pi}{p} = 2\pi \sqrt{\frac{N}{wl\Omega \cos \varphi}}.$$

For $\varphi = 45°$ and $N = 210$ in the Sperry, we have

$$p = 0.00123 \text{ rad. per sec.}, \qquad wl = 6.3 \text{ ft.-lbs.}$$

The damped period is somewhat longer, about 90 minutes; hence

$$\frac{2\pi}{q} = 90 \times 60, \qquad q = 0.00116 \text{ rad. per sec.}$$

From (6),

$$\frac{\mu^2}{4N^2} = (p + q)(p - q)$$

or

$$\mu = 0.16.$$

Hence from (11)

$$\sin \epsilon = 0.33, \qquad \cos \epsilon = 0.94, \qquad \epsilon = 19° = 0.33 \text{ rad.}$$

The ratio r of any γ amplitude (from $\gamma = 0$ to $\nu = 0$) to the previous amplitude on the *other side* of the ν axis—the time from $\gamma = 0$, to γ, and back to $\gamma = 0$ being *one-half* T, p. 136—is

$$r = e^{-(\mu\pi/2Nq)} = \frac{1}{3};$$

r is the *damping factor* and is more useful than the logarithmic decrement.

To find the ratio of any γ amplitude to the preceding ν amplitude let the axle swing from $\nu = \nu_0$ to $\nu = 0$; then

$$\gamma = 0, \quad \nu = \nu_0 \text{ at } t = 0, \quad \text{and} \quad \nu = 0 \text{ at } qt + \epsilon = \frac{\pi}{2},$$

whence, (10),

$$\gamma = \frac{b\nu_0}{Nq} e^{-(\mu/2Nq)[(\pi/2)-\epsilon]} \cos \epsilon$$
$$= 16\nu_0.$$

Since any ν for which $\gamma = 0$ may be taken as ν_0, any γ amplitude is 16 times the preceding ν amplitude. If there is no damping, $\mu = 0$, $\epsilon = 0°$, $\psi_\epsilon = 0$, § 75, whence

$$\gamma = \psi = 24\nu_0.$$

With the above values the equilibrium position of the *damped* motion is

$$\vartheta_e = 6 \textit{ minutes above the horizontal,}$$

$$\psi_e = 1.35 \textit{ degrees east of the meridian.}$$

Equations (10) and (12) are plotted in Fig. 60; notice the relative positions of maximum γ and zero ν. Curve I is tangent to the exponential curves at the points where $qt = \pi/2, \frac{3}{2}\pi$; curve II is tangent to the exponential curves at the points where $qt = \pi - \epsilon, 2\pi - \epsilon$.

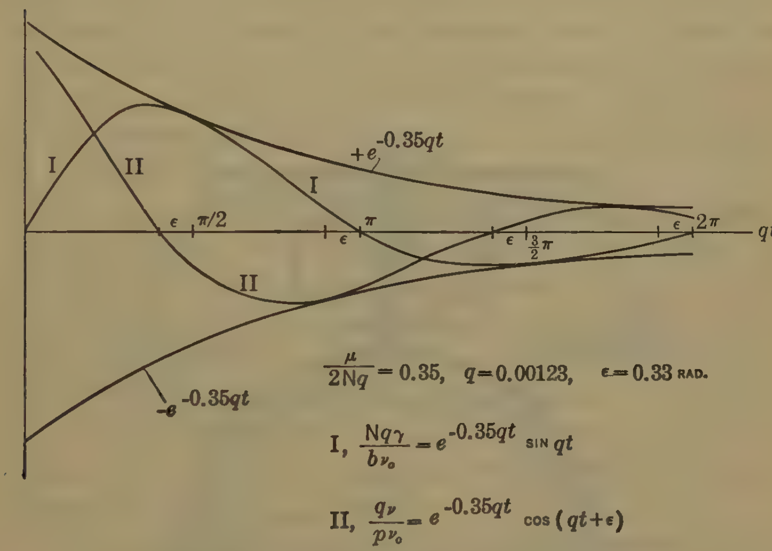


FIG. 60

Figure 61 on page 140 shows the locus of the north end of the axle; the ordinates ν are magnified 6 times. The largest value of γ is too large to satisfy the equation of small motion very closely; in practice, the initial tilt ν_0 is much less than a degree, and γ is correspondingly reduced. The graph (Fig. 61) shows that the axle will be near enough to its resting position some 2 hours after the start.

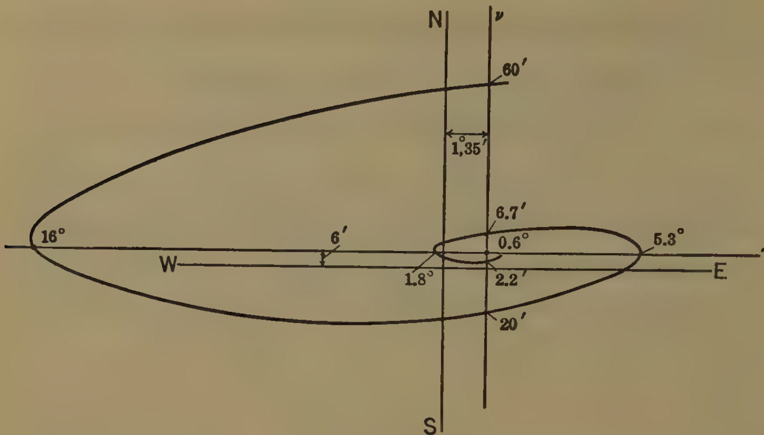
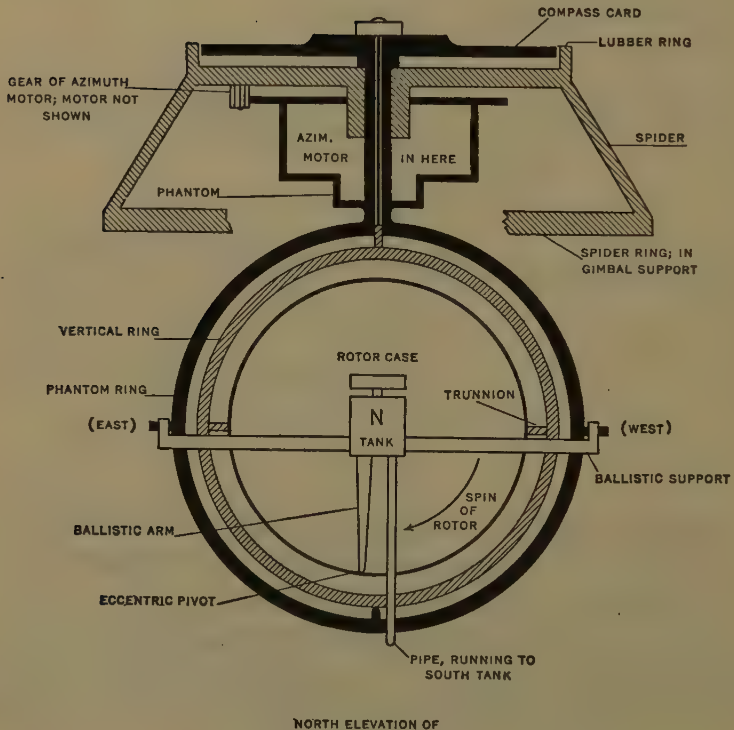


FIG. 61



NORTH ELEVATION OF SPERRY TYPE OF COMPASS

FIG. 62

79. The Sperry Compass. The principal features of the Sperry compass are illustrated in Fig. 62, the mechanical details of the actual apparatus pictured in the frontispiece being omitted or changed in order to make the diagram easy to read.

The axle of an electrically driven rotor is mounted horizontally in a case pivoted on horizontal trunnions carried by a vertical ring. This permits nutation of rotor and case about a horizontal axis through the trunnions.

The vertical ring is suspended from the head of the phantom by strands of steel wire. Attached to the vertical ring, north and south of its plane, are compensator masses to make the moments of inertia of case, rotor, and vertical ring equal about all axes in the plane of the supporting gimbals; these axes are thus principal, and rotation about them will not cause rotation of the suspended system about the suspension wire.

The compass card is fixed on the phantom, the lower part of which is the phantom ring. Vertical pivots merely keep the phantom ring and the vertical ring concentric without carrying any of the suspended weight. The rings are normally coplanar.

The mercury ballistic, Fig. 63, turns freely about trunnions on the phantom ring, their axis passing through the case trunnions. The ballistic arm attached to the ballistic frame is connected to a pivot placed eccentrically (east of the suspension axis) at the bottom of the case.

The spider is a frame supported in gimbals in a stand bolted to the deck of the ship. The lubber line on the spider is parallel to the ship's keel and indicates the course. The phantom rests on and turns in the spider.

Between the phantom and vertical rings are electric contactors (not shown) so devised as to pass current only when the two rings are out of plane with each other. The current operates the azimuth motor, which drives the phantom in azimuth (about the vertical) to keep the rings coplanar. The phantom thus follows the azimuthal motion of the rotor axle, prevents torsion

of the suspension wire, and maintains the ballistic in its proper position relative to the rotor case; for example, during the turning of the ship, the phantom is driven back at the same rate and therefore does not move relatively to the rotor case and the vertical ring.

The compass in the frontispiece is equipped with two ballistics for the sake of symmetry.

80. The Mercury Ballistic. If the gyro-pendulum of Fig. 57 is mounted aboard a moving ship, the acceleration of the origin produces moments L_0, M_0, N_0, § 20; the chief problem in the design of a gyro-compass is to prevent or to counteract the disturbing effects caused by them.

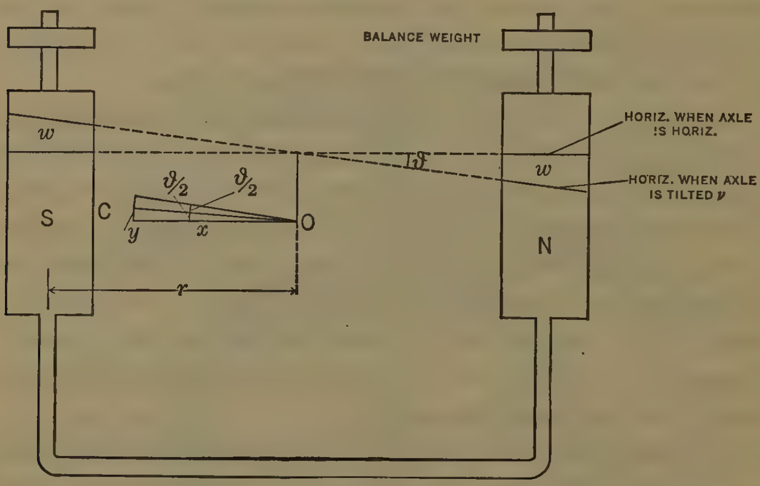


FIG. 63

In the latest types of the Sperry compass the gravity moment is supplied by the mercury ballistic,[1] Fig. 63. Two tanks, connected at their bottoms by a tube of about 0.12 inch diameter, are mounted on the transverse axis through O, the center of the rotor, so as to tilt with the rotor axle; the tanks and pipe are in the HV plane of Fig. 57, N being the north and S the south tank. Balance weights are attached to put the centroid

[1] Harrison and Rawlings, U. S. patent 1362940, 1920.

of the empty tanks and their attachments at O; then mercury is poured into the tanks until its centroid is also at O, that of the entire ballastic being at O. Generally, however, the mercury level passes through O and the weights are adjusted until the entire centroid is also at O. Figure 63 shows the mercury surface in two positions, *level in each:* first when the axle is horizontal, second when the N end of the axle is tilted ϑ radians above the horizontal.

Let W be the weight of all the mercury, and w the weight of that part of the mercury that flows from the N into the S tank. The flow of mercury shifts w through a distance $2r$ parallel to the original horizontal and lifts w a height $r\vartheta$ above it; this displaces the centroid of W from O to C; hence

$$2wr = Wx,$$
$$wr\vartheta = Wy,$$

and therefore

$$y/x = \vartheta/2.$$

In the Sperry Mark V compass, $r = 5.25$ inches and the section area of one tank is 2.8 sq. in. For a small tilt ϑ and a specific weight of 850 lbs. per cu. ft. of mercury, we have

$$Wx = 2wr = 2 \times 850 \times \frac{2.8}{144} \times \frac{5.25\vartheta}{12} \times \frac{5.25}{12}$$
$$= 6.3\vartheta \text{ ft.-lbs.},$$
$$Wy = wr\vartheta = 3.1\vartheta^2 \text{ ft.-lbs.}$$

Since ϑ is small—much less than a degree—the moment Wy and the distance y are negligible. *This is an important feature of the ballistic.*

The gravity moment $Wx = 6.3\vartheta$ ft.-lbs. corresponds to $wl\vartheta$ of the gyro-pendulum, *i.e.*, 6.3 to wl, but they are opposite in sense. Therefore if the ballistic is to produce right-handed precession $\dot{\psi}$, which is necessary for the axle to keep up with

the meridian plane, the Sperry rotor must spin left-handed viewed from the S end of the axle. This reverses the signs of a, b, N in (6) and (7), p. 132; stability, § 101, requires $\mu\nu$ to be reversed, *i.e.*, made right-handed in Fig. 57. This leaves all signs in the equations effectively unchanged.

The eccentric pivot is attached at an angle χ from the bottom point of the case in Fig. 62; χ is about 1°. The force exerted by the eccentric pin produces a vertical moment $Wx\chi$; *this is the damping moment.* Since $Wx = 6.3\vartheta$ the damping moment is

$$\mu\vartheta = 6\chi\vartheta;$$

hence

$$\mu = 0.11 \text{ ft.-lb. per degree of } \chi.$$

81. Stability of the Sperry. With the signs of a, b, μ, N made negative in (5), p. 133, the coefficient of λ^2 is

$$+ (N^2 - aJ - bI).$$

Unless

$$N^2 > aJ + bI$$

the motion is unstable even if $\mu = 0$ (see § 101); the case and vertical ring should therefore be light and the angular momentum large.

Except for this restriction, it might seem as if the Sperry and the gyro-pendulum differed only in the way the damping moment is applied. The Anschuetz was damped by the reaction of two horizontally westward flowing air jets expelled by the rotor, acting as a blower, through two small orifices near the bottom of the rotor case, one N and one S of the vertical axis. When the N end of the axle tilts upward, the N orifice is automatically enlarged and the S diminished by a pendulous plate or door; the unequal reactions of the jets then exert a vertical moment resisting the azimuthal swing. The compass is suspended from a hollow case floating in mercury; the viscous resistance to the

motion of the float slightly increases the damping moment and is to be imagined as included in μ. Equation (5), p. 132, therefore applies to the entire system suspended from the float.

Consider now the Sperry phantom and ballistic, with motor removed, as a free body:[1] the forces on it are the reactions of the mercury, the nearly horizontal forces of the case on the eccentric pin, the weight, the reaction of the spider, the pull of the suspension wire, and the horizontal forces exerted by the motor. Since the inertia of this system is negligible, the torque exerted by the motor balances the vertical moment $\mu\vartheta$ exerted by the pin. The motor therefore applies the damping moment. Consequently (5), p. 132, describes also the behavior of the Sperry compass. But it is shown in § 89 that the viscosity of the mercury introduces a positive λ^3 term into the equation. The compass will be damped and stable if the coefficients are made to satisfy Routh's conditions, p. 172. Although damping does not occur when $\mu = 0$ (pin central) it is partly due to the *dissipation* of energy in the ballistic; the case is somewhat like that of the rising top. The motor continually puts energy into the system; at least *some* of it is used in lifting the mercury centroid, which is slightly lowered when the ballistic is tilted. The energy dissipated through viscosity comes from the kinetic energy of the moving parts. Damping therefore does not seem to be possible without some viscosity. But it is only the fast tremors, p. 129, that are extinguished by friction.

82. Route Errors. The route errors of a ship's compass are produced by changes of course and speed. The effects of rolling, pitching and vibration are taken up later. Let the *east* and *north* velocities of the ship be u, v respectively, and let U, V be the corresponding accelerations. By Theorem V, p. 8, the accelerations of the origin in Fig. 57 are, if R is the earth's radius

[1] The term *free body* was introduced into the teaching of mechanics by I. P. Church in his *Mechanics of Engineering* (First Edition, 1890). In a letter to the author Professor Church stated that the first mention of it occurs in Coxe's translation of Weisbach, *Mechanics of Engineering* (1870).

(6357×10^3 to 6378×10^3 meters for the International Ellipsoid),

v^2/R toward center: negligible,

V horizontally northward,

$2v\Omega \sin \varphi$ horizontally westward,

u^2/R toward center: negligible,

U horizontally eastward,

$2u\Omega \sin \varphi$ horizontally northward.

The effect of the eastward (or westward) velocity of the ship ought also to be taken into account. It produces a virtual increase (or decrease) of the earth's spin, which, however, is negligible except in very high latitudes. Another effect of the ship's motion needs to be considered. The pull of the earth constrains the north end of the gyro axle to dip with the bow of the ship in its northward motion, *i.e.* the horizontal plane approaches perpendicularity to the earth's axis. The XYZ axes of Fig. 57 thus turn about Z, when ψ is small, with a velocity $-v/R = -\dot{\varphi}$ from Y to X. This motion is superposed on $\dot{\vartheta}$, making the nutation of the axle $\dot{\vartheta} - \dot{\varphi}$ provided Z points nearly due east.

As the centroid of the mercury in the ballistic remains on a horizontal through O, y being negligible, only the east and west accelerations enter the terms L_0, M_0, N_0, p. 32; hence

$$M_0 = \frac{Wx}{g}(U - 2v\Omega \sin \varphi).$$

But $Wx = b\vartheta$, where $b = 6.3$ ft.-lbs., p. 143. Hence

$$M_0 = \left(\frac{bU}{g} - \frac{2bv\Omega \sin \varphi}{g}\right)\vartheta$$

$$= k\vartheta, \text{ say.} \tag{1}$$

The north acceleration V tilts or banks the mercury surface at an angle V/g to the horizontal, raising the mercury in the south tank and lowering it in the north. This increases the angle of the mercury wedges from ϑ to $\vartheta + (V/g)$, thereby

increasing the gravity and damping moments by $b(V/g)$ and $\mu(V/g)$, respectively; *M_0 is likewise increased by $k(V/g)$.*

With all signs changed—*i.e.* with no signs changed—(1), (2), p. 131, define the motion of the Sperry compass. To include M_0, note that in (1), p. 32, the external or applied moments and M_0 are on opposite sides of the equation; consequently $-M_0$ should be added to the left number of (2), p. 131. With the modification due to acceleration the equations become

$$(\mu - k)\vartheta + N(\dot{\nu} - \dot{\varphi}) + a\psi - k\frac{V}{g} = 0, \tag{2}$$

$$b\frac{V}{g} - N\dot{\psi} + b\nu = 0. \tag{3}$$

83. Values of k. Taking 25 knots as the greatest speed of vessels on which the gyro-compass is used, we have, since 1 knot = 6080 ft. per hour,

greatest velocity = 25 knots = 42 ft./sec.

A battleship cannot come to speed at constant acceleration in less than about 4 minutes; in this case

greatest acceleration = 0.17 ft./sec.^{2}

The acceleration of a ship steaming in a circle at 25 knots is

17.6 ft./sec.^{2} *per* 1000-*foot radius.*

The greatest value of $(2b/g)v\Omega \sin \varphi$ is

0.0008 *at* $\varphi = 45°$

and is negligible. Hence

$$k = \frac{bU}{g},$$

where U is the acceleration toward the east.

For steaming along a meridian $k = 0$; along a latitude circle $k = 0.005b$ and is negligible. For radial acceleration along a meridian $k = 0$; along a latitude circle $k = 0.5b$ per 1000-ft.

radius. In all of these cases $k(V/g)$ is either zero or negligible; the largest value of k in the first term of (2) is $0.5b$; when the ship is headed north (or south), is getting up speed and is turning, the term $k(V/g)$ may be appreciable.

84. Resting Position. The values of ψ and ϑ for which $\dot{\psi} = 0$ and $\dot{\vartheta} = 0$, give the resting position of the gyro axle. Calling them ψ_r and ϑ_r, we get from (2) and (3), p. 147, with $k = (bU/g)$,

$$(1) \qquad \vartheta_r = \vartheta_\epsilon - \frac{V}{g}.$$

Substituting this in ψ_r, and using the values of ϑ_ϵ and ψ_ϵ from p. 131, we find

$$(2) \quad \psi_r = -\frac{\mu}{b}\tan\varphi + \frac{\dot{\varphi}}{\Omega\cos\varphi} + \frac{\mu V}{gAn\Omega\cos\varphi} + \frac{U}{g}\tan\varphi.$$

The resting position changes not only with the latitude but also with the velocity $\dot{\varphi}$. The term is therefore misleading: what is meant is that the resting position is the equilibrium position for the kinetic reactions produced by the motion of the ship.

The route errors are found from (1), (2):

north velocity $\dot{\varphi}$ deflects the axle $\dot{\varphi}/(\Omega\cos\varphi)$ radians west of ψ_ϵ;

north acceleration V decreases the tilt by V/g radians and turns the axle $\mu V/(gAn\Omega\cos\varphi)$ radians west of ψ_ϵ;

east acceleration U deflects the axle $(U/g)\tan\varphi$ radians west of the meridian.

In § 69 it was assumed that the suspension axis stays in the meridian. If it is deflected east or west of the vertical, the ballistic and damping moments change; see § 94.

85. Rectilinear North Acceleration. In this case $k = 0$ and $V = R\ddot{\varphi}$, where R is the earth's radius; (2), (3), p. 147, become

$$(a) \qquad \mu\dot{\vartheta} + N(\dot{\nu} - \dot{\varphi}) + a\psi = 0,$$

$$(b) \qquad \frac{bR}{g}\ddot{\varphi} - N\dot{\psi} + b\nu = 0,$$

or, in terms of γ and ν, as in § 75,

(1) $$\mu\nu + N\dot{\nu} - N\dot{\varphi} + a\gamma = 0,$$

(2) $$\frac{bR}{g}\ddot{\varphi} - N\dot{\gamma} + b\nu = 0.$$

To prepare these for integration, change the variable from γ to δ by means of

(3) $$-N\dot{\varphi} + a\gamma = a\delta;$$

then

(4) $$\mu\nu + N\dot{\nu} + a\delta = 0,$$

(5) $$\left(\frac{bR}{g} - \frac{N^2}{a}\right)\ddot{\varphi} - N\dot{\delta} + b\nu = 0.$$

To simplify (5) put

$$\frac{bR}{g} - \frac{N^2}{a} = 0;$$

from (3), p. 135,

(6) $$p^2 = \frac{g}{R},$$

whence the free period of the compass will be

$$T = 2\pi\sqrt{\frac{R}{g}},$$

$$= 84.5 \text{ minutes } \textit{at the equator},$$

for $R = 637.8 \times 10^4$ meters, $g = 9.78$ meters/sec.^{2}; see also § 86.

It is stated by writers on the gyro-compass that when the free period is 84.5 minutes, the compass moves aperiodically from one resting position to another. This cannot be exactly true because R depends on the latitude of the ship. To see whether it is true for constant meridian acceleration on a spherical earth, differentiate (2), p. 147, and put $\psi = 0$, $\vartheta = 0$; this gives

$$\ddot{\nu} = \ddot{\varphi},$$

which shows that the axle is not in equilibrium in what we have

called the resting position. The behavior of the axle in any position is found by integrating (4), (5) under condition (6). The equations are of type (6), (7), p. 132, and are integrated as in § 77. The general solution is

(7) $$\nu = e^{-(\mu t/2N)}(c_1 \cos qt + ic_2 \sin qt),$$

(8) $$\delta = e^{-(\mu t/2N)}(c_3 \cos qt + ic_4 \sin qt).$$

Proceeding exactly as before, but using the boundary conditions

$$\nu = \nu_0, \qquad \delta = \delta_0 \quad \text{when} \quad t = 0,$$

we get

$$\nu = \nu_0 e^{-(\mu t/2N)}\left(\cos qt - \frac{N\delta_0}{b\nu_0}(c + q) \sin qt\right),$$

$$\delta = \delta_0 e^{-(\mu t/2N)}(\cos qt + c \sin qt),$$

where

$$c = \frac{\mu}{2Nq} + \frac{b\nu_0}{Nq\delta_0}.$$

If the axle starts at the resting position, (1) and (2), p. 148, with $U = 0$ for a meridian course, and $\dot{\varphi} = \dot{\varphi}_0$, give

$$\nu_0 = \vartheta_r - \vartheta_e = -\frac{V}{g},$$

$$\delta_0 = \psi_r - \psi_e - \frac{N}{a}\dot{\varphi}_0 = \frac{\mu V}{ag}.$$

Hence

(9) $$\psi = \psi_e + \frac{N}{a}\dot{\varphi} + \frac{\mu V}{ag} e^{-(\mu t/2N)}\left\{\cos qt + \left(\frac{\mu}{2Nq} - \frac{ab}{\mu Nq}\right)\sin qt\right\}.$$

Since the coefficient of $\mu V/(ag)$ is never as large as unity, the axle lags behind its successive resting positions and the value of ψ_r given by (2), p. 148, is the greatest azimuthal deviation.

86. Ballistic Deflection. The nutation of the axle is slow, and if changes of ϑ and $\dot{\nu}$ can be disregarded in (*a*), p. 148,

$$\Delta\psi = \frac{N}{a}\Delta\dot{\varphi},$$

where $\Delta\psi$ is the difference between two successive *velocity* errors.

But the meridian *acceleration* V produces a separate independent change of ψ, which is found from (b), p. 148, to be, *if we neglect* bv, and put $\dot\psi = \Delta\psi/\Delta t$,

$$\Delta\psi = \frac{bV}{gN}\Delta t.$$

If this change is different from the first, the axle will be deflected from the resting position corresponding to the new velocity $\dot\varphi + \Delta\dot\varphi$; or if V is due to turning while on an east-west course the axle will be deflected from its normal position. This deflection is called *ballistic* because it is the *throw* of the axle caused by the acceleration of the ship.

For rectilinear acceleration $V = R\ddot\varphi = R(\Delta\dot\varphi/\Delta t)$ the compass can be designed so as to make the ballistic deflection equal to the change of velocity error; then

$$\frac{N^2}{a} = \frac{bR}{g}$$

as on p. 149. This is the interpretation of the 85-minute period, but it is only approximately correct as it neglects the changes of ϑ and $\dot\vartheta$. The last term of (9), p. 150, shows that the error thus made depends on the damping coefficient μ.

When the ballistic deflection is annulled or compensated for in this way, the gyro axle tilts at the rate at which the latitude is changing and remains normal (except for ϑ_ϵ) to the rotating radius of the earth. The expression $T = 2\pi\sqrt{R/g}$ is the period of a simple pendulum having its bob at the center of the earth; if R always remains parallel to the instantaneous radius of the earth it must follow that the acceleration of the point of support does not disturb the bob. Schüler[1] has pointed out that this is true also when R is the equivalent length of a compound pendulum; the interpretation however is purely formal because the earth does not attract a long pendulum, or one with its bob inside the earth, as it does a short one on the surface.

[1] Physikalische Zeitschrift, 1923, p. 344.

87. East-West Acceleration. For motion along a parallel of latitude, $V = 0$, $\dot{\varphi} = 0$, $k = (b/g)U$ in (2), (3), p. 147. If we put

$$\mu - (b/g)U = \mu',$$

the equations are like those on p. 132. The effect of U is to decrease the damping when the acceleration is eastward and to increase it when westward. If U is large enough to make μ' negative the motion is unstable. This case is likely to occur when the ship makes a sharp turn at high speed or when it is headed north and rolls with sufficient violence. See § 94.

88. General Motion of the Mercury. It has thus far been assumed that the mercury stays level during the settling motion of the compass when the ship is at rest, and that its centroid shifts promptly in response to the ship's acceleration. This is possible only for zero viscosity, in which case, however, oscillations of the mercury, once started, persist and disturb the pointing of the needle. As is shown in § 81, some viscosity is required for damping and some also for the elimination of serious errors caused by the rolling and pitching of the ship. The extreme case of infinite viscosity obviously makes the ballistic inoperative.

The ship's motion accelerates the suspended element. If the mercury centroid is not central, the E-W acceleration component acts on the shifted centroid and exerts a moment about the vertical suspension axis; this moment deviates the compass and is the cause of the rolling error. When, as we shall suppose, the rotor axle is in or very near the meridian plane, the N or S acceleration component, A in Fig. 64, displaces the mercury centroid S or N. To find the effect of A, modify the gravitational field of force (see § 16) by combining g with *reversed* A. As in the case of a basin of water on an accelerating train, the new or virtual 'horizontal' is normal to the resultant of g and $-A$; the acceleration in the modified field is $(g^2 + A^2)^{1/2}$, which in compass problems is not different enough from g to be taken into account so far as *weight* is concerned.

All angles in Fig. 64 are small, the cosines may be taken as unity and the sines may be taken equal to the angles. Let

M = mass of mercury in tanks excluding tubes;

m = mass of mercury in tube; M and m are constant;

ρ = density of mercury;

u = relative velocity in tube;

v = absolute velocity in tank;

A, a = section areas of one tank and tube;

L = average depth of mercury in tank;

$2l$ = horizontal length of tube, vertical parts neglected;

c = loss of energy per lb. per unit velocity u in tube; this is due to viscosity and bends—loss in tanks is neglected.

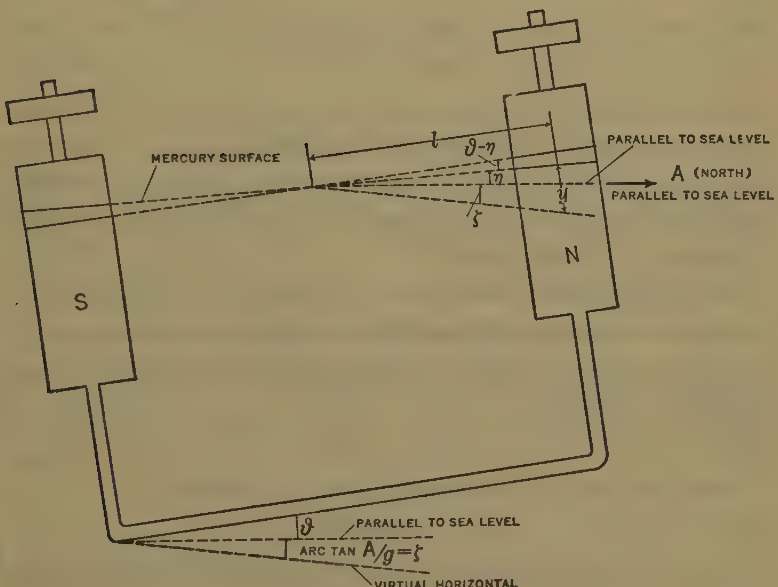


FIG. 64

The velocity u produces a relative velocity $l(\dot{\vartheta} - \dot{\eta})$ in the tanks. Flow occurs even when η is constant.

From the diagram

$$v = -l\dot{\eta}, \qquad au = Al(\dot{\vartheta} - \dot{\eta}), \qquad y = l(\eta + \zeta).$$

In the value of v the motion of the ballistic itself is disregarded.

In time dt an element of mass dM falls a distance $2y$ from the surface in the N tank to that in the S tank; y is perpendicular to sea level. The energy equation is

$$(2y - cu)gdM = \frac{d}{2}(Mv^2 + mu^2),$$

where $dM = \rho Avdt$, but M and m are constant. Putting

$$m = 2al\rho, \qquad M = 2AL\rho,$$

and noting that A/a is large and $\dot{\vartheta}/\dot{\eta}$ is small, we get

(1) $$h(\ddot{\eta} - \ddot{\vartheta}) + k(\dot{\eta} - \dot{\vartheta}) + \eta = -\zeta,$$

where

$$h = \frac{Al}{ag}, \qquad k = \frac{cA}{2a}.$$

Since k in (1) corresponds to k in (8), p. 134, the viscous resistance varies as the velocity through the tube. From (2), below, the velocity of fall of head of mercury varies as the head; this is probably not strictly correct, but it is at least a first approximation to the truth.

89. Ship at Rest. An idea of the effect of viscosity on the settling motion of the mercury when the ship is at rest or going at constant velocity is found by neglecting the first term of (1) and putting $\zeta = 0$; then

(2) $$k(\dot{\vartheta} - \dot{\eta}) = \eta.$$

When the ballistic is at rest, $k\dot{\eta} = -\eta$; hence

(3) $$\eta = \eta_0 e^{-(t/k)},$$

η_0 being the initial head angle. Experiments on the Sperry Mark V ballistic gave $k = 27$.

The ballistic and damping moments, depending on the excess liquid in the lower tank, are

$$b(\vartheta - \eta) \quad \text{and} \quad \mu(\vartheta - \eta).$$

With these values (1) and (2), p. 131, become

$$\mu(\vartheta - \eta) + N\dot{\nu} + a\dot{\psi} = 0, \tag{4}$$

$$b(\vartheta - \eta) - N\dot{\psi} - b\vartheta_e = 0. \tag{5}$$

The result of eliminating η, ψ, and their derivatives from (2)–(5) and their derivatives is

$$kN^2\dddot{\nu} + N^2\ddot{\nu} + \mu N\dot{\nu} + ab\nu = 0, \tag{6}$$

whence, if

$$\nu = e^{\lambda t},$$

$$kN^2\lambda^3 + N^2\lambda^2 + \mu N\lambda + ab = 0. \tag{7}$$

But observations show that the behavior of the Sperry is quite accurately defined, as in § 77, by

$$N^2\lambda^2 + \mu N\lambda + ab = 0. \tag{8}$$

Consequently the roots of (8) cannot be much different from two of the roots of (7), and the third root of (7) is therefore nearly

$$-\frac{ab}{kN^2}\Big/\frac{ab}{N^2} = -\frac{1}{k} = -\frac{1}{27}.$$

We may also proceed as follows. Divide (7) by N^2, making the last term p^2, § 77, where $p = 0.00123$; then with $N = 210$ and $\mu = 0.16$, (7) reduces to

$$27\lambda^3 + \lambda^2 + 0.0008\lambda + 0.0000 = 0,$$

which is negative for -0.04 and positive for -0.03. Note that $1/27$ lies between these values. The solution of (7) therefore consists of

$$\nu = ce^{-(t/k)}$$

superposed on the solution of (8). In 4.5 minutes

$$e^{-(t/27)} = 0.000045,$$

which shows that when k is small the exponential term soon vanishes and (8) is accurate enough for the Sperry compass.

Ex. 130. Show that when I, J, and the first term of (1), p. 154, are not neglected the equation corresponding to (7), p. 155, is of the sixth order; take $\zeta = 0$.

Ex. 131. Show that for stability $k < 300$ in (7) for $\mu = 0.10$; see § 101.

When the viscosity is too large to be disregarded, the displacement of the mercury centroid is so slow as to decrease the effectiveness of the 85-minute period in compensating for the ballistic deflection. Although some viscosity is necessary for damping, the large motions are damped practically only by the eccentricity. However, the viscous drag in the ballistic tube decreases the b and μ moments by a moment proportional to u, say $\lambda(\dot{\vartheta} - \dot{\eta})$. This ought to have been included in the equations of motion, but its omission is not serious in view of the rather ruthless approximations we have made.

90. The Rolling Moment. One of the important problems in gyro-compass design is that of reducing the deviation of the needle (rotor axle) caused by the rolling of the ship. Pitching is understood to be included in rolling. Rolling will be taken as simple harmonic motion of period equal to that of the ship itself. Waves that come regularly for a sufficient time impress their own period on that of the ship, but their action is generally so erratic that they can be looked upon as a series of intermittent disturbances lasting too short a time to force oscillation on the ship. The ship never completely forsakes its own period.

A gyro-compass has an enormous dynamical moment of inertia about its E-W axis (see p. 134). To a first approximation, the small alternating N-S acceleration of the rolling is without influence on the tilt of the rotor axle, even when the compass is pendulous.

The gyro-pendulum or old type Anschuetz, being free to swing about a N-S axis, is a short-period pendulum swinging in synchronism with the ship. In Fig. 65, C is the center of the rotor; E, W are the respective positions of the centroid when the ship has rolled to the extreme right (port) and extreme left

(starboard). R, in each case opposite to the acceleration of the compass and ship, is the kinetic reaction due to it. When E moves to W, R decreases to zero at C, then reverses and reaches its maximum at W. On a meridian course R always passes through C; on an EW course the centroid stays at C. If the compass is freely suspended the moment of the reactions R produces an azimuthal deflection called the rolling error, and also a tilt of the axle. See § 94 for a further discussion.

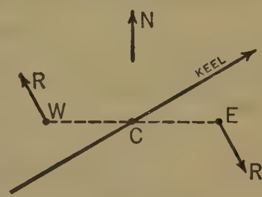


FIG. 65

Ex. 132. Prove that the moment of R is greatest on a 45° course.

91. The Tri-Gyro-Compass. The rolling moment, arising from the E-W shift of the centroid synchronizing with the period of the ship, will be materially reduced if the E-W period of the compass —swinging as a pendulum—can be lengthened to keep the centroid *on the same side for a large number of rolls.* In this case the moment alternates and has no cumulative effect. The necessary lengthening of period can be got by increasing the dynamical (not statical) moment of inertia about the N-S axis gyroscopically. This is done in the Anschuetz and Arma tri-gyro-compasses by mounting two supplementary gyros on the frame carrying the main rotor (Fig. 66). The middle gyro is not necessary. It is absent in recent models of the Arma, the two remaining gyros being mounted with their centers on the NS axis.

As will be evident presently, the stabilizing gyros must have freedom to precess about their vertical axes. They are connected by springs and links (not shown) to limit their motion, to make them move symmetrically with respect to NS, and to hold them normally at 30° to NS.

Let y be the *small* angle through which the frame turns about NS, W end up, x the angle shown, w the weight of the system, l the distance of the centroid below the point of support (center

of frame and coplanar with the centers of the gyros). Each gyro has angular momentum N. Let $\theta = (\pi/6) - x$ radians.

The E-W momentum $N \sin\theta$ of B turns clockwise at $\dot{x}$, requires a moment $(N \sin\theta)\dot{x}$ right-handed about NS to maintain the

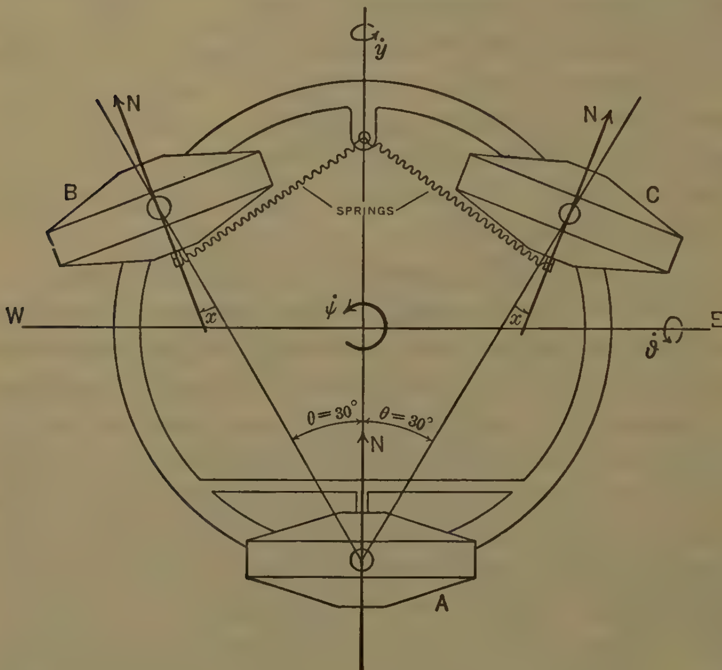


Fig. 66

precession $\dot{x}$, and therefore exerts $-N\dot{x}\sin\theta$ on the frame. C exerts an equal moment, whence if A' is the statical moment of inertia of the whole system about NS

$$(1) \qquad -wly - 2N\dot{x}\sin\theta = A'\ddot{y},$$

where $\theta = 30°$ to the first order of small quantities.

The $\ddot{x}$ equation of one gyro, say B, will serve to eliminate $\dot{x}$ from (1). Let s be the vertical moment exerted by the spring for $x = 1$ radian; then sx is the moment for the deviation x; it opposes x and $\dot{x}$.

The E-W momentum $N \sin\theta$ of B is rotated right-handed about the northward horizontal by $\dot{y}$ and by the earth's rotation $\Omega \cos\varphi$ in latitude φ. The resulting vector change-rate of momentum is

$$N\dot{y} \sin\theta + N\Omega \sin\theta \cos\varphi.$$

Hence if C' is the moment of inertia of the B gyro about a vertical normal to its axle,

$$(2) \qquad sx = -C'\ddot{x} + N\dot{y}\sin\theta + N\Omega \sin\theta \cos\varphi.$$

Neglecting the statical moments of inertia in (1) and (2), eliminating $\dot{x}$, noting that $\dot{\theta} = -\dot{x}$, and rejecting $\dot{x}\dot{y}$, we get

$$\frac{2N^2 \sin^2\theta}{s + N\Omega \cos\theta \cos\varphi}\ddot{y} = -wly,$$

which is simple harmonic motion of angular frequency given by

$$\omega^2 = \frac{wl(s + N\Omega \cos\theta \cos\varphi)}{2N^2 \sin^2\theta}.$$

In the coefficient of $\ddot{y}$, the dynamical moment of inertia, s is to be chosen to make ω small and to keep the amplitude x within a few degrees so as not to disturb the NS angular momentum. When the period of the lateral swings is of the order of magnitude of the north-south swings, any acceleration that produces lateral oscillations will also start the other; both motions arise from the same cause and will thus be in the same phase. Consequently if the dynamical moments of inertia about the two horizontal axes are equal, the periods are equal, the centroid oscillates in a vertical 45-degree plane through the center (point of support), and the rolling moment is zero. In other words, R in Fig. 65 always passes through C. On account of the difficulty of getting a spring weak enough to give ω its proper value and to keep x small, it is not possible to have the periods equal. Nevertheless most of the rolling moment can be eliminated by a proper compromise.

92. Horizontal Damping. The Anschuetz tri-gyro-compass (in the most recent model the middle gyro is omitted) is damped by means of the flow of oil in a horizontal annulus (a ring-shaped tube) mounted on the rotor case. As it tilts with the rotor the centroid of the oil moves toward the lower side of the ring and opposes the ballistic moment but lags behind it on account of viscosity.[1] The Arma and the Brown compasses are also damped by horizontal moments.

To find how the damping moment should vary with the tilt, omit $\mu\nu$ from (1), p. 131, and introduce a horizontal moment $f(\nu)$; then

$$N\dot{\nu} + a\psi = 0,$$

$$-N\dot{\psi} + b\nu + f(\nu) = 0.$$

Hence

(1) $$N^2\ddot{\nu} + ab\nu + af(\nu) = 0.$$

If (1) is to represent damped oscillations, we must have

(2) $$f(\nu) = c\dot{\nu},$$

c being a suitable constant. Equation (2) means that the damping moment varies as the *velocity* of tilting. But if the tilt has the typical form

$$\nu = e^{-kt}\sin\omega t,$$

it follows that

$$f(\nu) = c\dot{\nu} = ce^{-kt}\sin(\omega t - \epsilon);$$

hence the motion of the oil must be damped oscillatory, lagging behind that of the rotor axle. Note also the effect of k, § 89.

93. Forced Oscillations of the Mercury. Before discussing the reduction of the rolling moment in the Sperry compass we shall study the motion of the mercury when the ship rolls with simple harmonic motion of angular frequency ω. In (1), p. 154, take

$$\zeta = -\zeta_0 \sin\omega t,$$

[1] For a study of the flow in the annulus see Béghin, Comptes rendus, 1921, p. 288; also Annales hydrographiques, 1921, *Étude théorique des compas gyro-statiques.*

where the minus sign is introduced for the convenience of making the right member positive. The equation of motion is then, if the ballistic is *at rest* ($\dot{\vartheta} = \ddot{\vartheta} = 0$),

(1) $$\ddot{\eta} + 2m\dot{\eta} + n^2\eta = r \sin \omega t,$$

where

$$m = \frac{k}{2h}, \qquad n^2 = \frac{1}{h}, \qquad r = \frac{\zeta_0}{h},$$

which can be written

(2) $$\ddot{\eta}_1 + 2m\dot{\eta}_1 + n^2\eta_1 = 0,$$

(3) $$\ddot{\eta}_2 + 2m\dot{\eta}_2 + n^2\eta_2 = r \sin \omega t,$$

where

$$\eta_1 + \eta_2 = \eta.$$

The solution of (2) gives the *free* damped motion—free because there is no external force; n is the *natural* angular frequency for $m = 0$.

The solution of (3), giving the *forced* oscillations induced by the disturbing force, is

$$\eta_2 = R \sin (\omega t - \epsilon),$$

where

$$\tan \epsilon = \frac{2m\omega}{n^2 - \omega^2},$$

(4) $$R = \frac{r \sin \epsilon}{2m\omega}.$$

The pendulous swings, τ, of the suspended element in the gimbals are defined by an equation of the form (1), say

$$\ddot{\tau} + 2c\dot{\tau} + p^2\tau = s \sin \omega t,$$

the forced swings being

(5) $$\tau = \tau_0 \sin (\omega t - \delta),$$

where

$$\tan \delta = \frac{2c\omega}{p^2 - \omega^2}.$$

94. The Rolling Moment. The suspension and follow-up mechanism of the Sperry does not permit vertical moments applied to the *ballistic* to deflect the compass. They are resisted by the azimuth motor, which is able to resist as hard a horizontal *E-W* push as one can apply by hand to the *ballistic.* Consequently, if the *E-W* trunnion axis were kept *horizontal* (why, will be seen presently), the rolling of the ship would produce no error of the needle, all *E-W* reactions on the ballistic being resisted by the motor.

It was shown above that in the Anschuetz the error is practically eliminated by making the tilt very slow and thereby also keeping it small; but no azimuth motor is necessary in this case, although one is used to overcome the resistance of the mercury-float suspension. In the Anschuetz, the free period with which the ballistic moment alternates is about 90 minutes, as in all gyro-compasses; since the ship's oscillations are relatively fast, the amplitude of the forced oscillations is negligibly small; see (4), above. Therefore if the centroid (the point corresponding to the centroid in Fig. 54) stays east of the meridian about as long as it stays south, the alternating reactions due to the ship's motion practically balance out. However, as the period of the mercury in the Sperry ballistic is of the order of magnitude of the ship's period, the east reaction might occur when the centroid is south, and the west when the centroid is north. It is only the azimuth motor that resists the resulting moment.

To study the effect of the *E-W* swings of the Sperry, suppose the compass to swing until the east trunnion is tilted upward through an angle τ. Since the phantom constrains the ballistic to turn about the tilted axis—as if it were a door on an oblique hinge axis—the moment about it is

$$wl\vartheta \cos \tau,$$

where ϑ is either the meridian tilt of the ballistic or the angle

equivalent to a forced shift of the mercury centroid. The other component, $wl\vartheta \sin \tau$, is resisted by the phantom and *does not act on the rotor case;* but $wl\vartheta \cos \tau$ does act on the case and is the cause of rolling error. Similarly the damping component $\mu\vartheta \cos \tau$ about the inclined suspension axis is another source of error.

It follows that ballistic and damping moments, regardless of how or by what mechanism they are applied, cause rolling error on account of the *E-W* swings of the compass unless some method of compensation or elimination can be found. This is shown in Fig. 67, which represents the trunnion and suspension

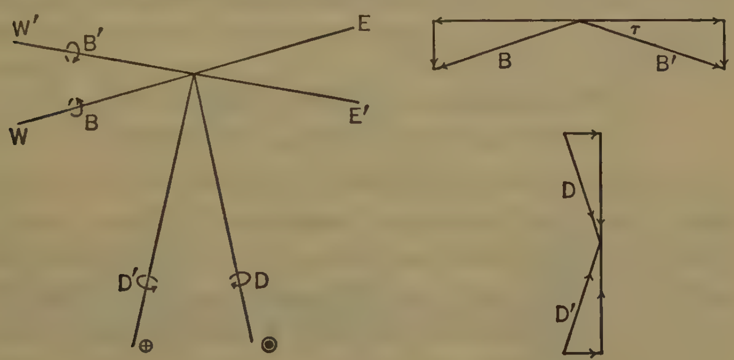


FIG. 67

axes viewed from the south. A dot in the center of the eccentric-pin circle denotes an arrow (the force at the pin) pointing toward the reader; a cross shows the tail of the arrow.

The maximum ballistic moments B and B' are opposite, and the damping moments D and D' are opposite, when the forced oscillations of the mercury are in phase with—or opposite in phase to—the *E-W* pendulous swings. This is the most unfavorable case because the maximum moments occur at the maximum tilts τ. The vertical components of B and B' and the horizontal components of D and D' are the disturbing or rolling moments acting on the case.

However, the motions of the mercury and the pendulum are

not in the same or opposite phases, because the mercury motion is damped by viscosity and the pendulous swings by dashpots on the gimbal rings. The choice of phase difference is largely a matter of design. After the free oscillations are damped out, both systems have the angular frequency ω of the ship. If we neglect the phase difference between pendulum and ship, we have from (5), p. 161,

$$\tau = \tau_0 \sin \omega t,$$

and similarly, if ϑ_0 is the amplitude of the mercury centroid,

$\vartheta = \vartheta_0 \sin \omega t$ *for mercury and pendulum in same phase,*

$\vartheta = \vartheta_0 \cos \omega t$ *for mercury and pendulum* 90° *different in phase.*

The ballistic moment is then

$wl\vartheta_0\tau_0 \sin^2 \omega t$ *for zero phase difference,*

$wl\vartheta_0\tau_0 \sin \omega t \cos \omega t$ *for* 90° *phase difference.*

The former is always positive; the latter has an average value of zero and its effect is therefore nil. Hence the more closely the phase difference between mercury and pendulum approaches 90° the more nearly is the rolling moment eliminated. An idea of the results attainable in practice is given by the following laboratory tests of a Sperry Mark X (No. 10001) made on a Scoresby frame that imitates the motions of a ship.

Total roll 80°, period 8 to 9 secs.; total pitch 20°, period 6 to 7 secs.; total yaw 15° to 25°, period irregular.

48-hour run with various alternations and combinations of roll, pitch, and yaw; also changes of headings.

Specified allowance: total azimuth variation 2°, maximum error 2°.

Test results: total azimuth variation 0.77°, maximum error 0.75°.

Ex. 133. Find the average ballistic rolling moment during $T = 2\pi/\omega$ secs. for a phase difference of ϵ, and show that it is greatest on a 45° course.

If the course is θ° west, $\vartheta_0 = c \sin \theta$, $\tau_0 = k \cos \theta$, where c and k are constants; hence

$$\vartheta = c \sin \theta \sin (\omega t + \epsilon), \qquad \tau = k \cos \theta \sin \omega t.$$

In Fig. 67 the horizontal components of B alternate and balance out; the vertical components are cumulative and are approximately equal to $wl\vartheta\tau$ since τ is small. Integrate this from 0 to 2π and divide by T.

Note that the factor c depends, from (4), p. 161, on $r \sin \epsilon/2m\omega$.

For a detailed discussion of this question see Rawlings, *Gyroscopic Compass and its Deviations.*

95. Automatic Steering. In addition to its freedom from several important defects of the magnetic compass—declination error, the shielding action of submarine hulls, and the errors produced on a battleship by the motion of large masses of steel—the gyro-compass has the advantage of permitting automatic steering. For this purpose electric contacts are so arranged that when the ship leaves its course the relative motion between the gyro-compass and the lubber line operates a relay that actuates a rudder-control or steering motor. This is the device known as *Metal Mike.* The results of different methods of steering are shown in Fig. 68; they speak for themselves.

FIG. 68

CHAPTER X

STABILIZERS

96. Gyro-Stabilizers. Gyro-stabilizers reduce the rolling of vessels and the deviation of torpedoes, and keep monorail vehicles upright. A ship must roll in order to operate the stabilizer; it functions like an engine governor in that it is actuated by the very motion it controls. A monorail carriage must be always in a state of oscillation. In both cases—vessel and carriage—the momentum axis of the gyro must be allowed to change its direction in order that the precession may produce the control moment. This chapter discusses principles rather than specific designs.

97. Schlick Stabilizer. The first operative stabilizer was invented by Schlick;[1] its mode of action is shown in Fig. 69. A heavy gyro, i, about 5 per cent. of the weight of the ship, spins about a vertical axis, is mounted on trunnions athwart the ship, and is counterweighted to be gravitationally stable. The rolling of the ship makes the axle nod fore and aft; the gyro then exerts a moment against the rolling of the ship.

In ii, the ship and the gyro have a starboard roll ψ; the gyro axle is tilted φ from the vertical. X, Y, Z are left-handed because most of the vectors are left-handed: the axes will not be used because we are not going to substitute in equations.

At the instant pictured, the momentum Cn of the rotor turns at the rate $\dot{\psi}$ and precesses at $\dot{\varphi}$. The resulting change-rate $Cn\dot{\varphi}$ requires for its maintenance a moment M about Y; $Cn\dot{\varphi}$ is not the only change-rate, and M is not the only moment about OY or OA; see (2), below. The gyro reacts in the opposite sense on the ship and thus opposes the rolling *motion* $\dot{\psi}$ of the

[1] Transactions of the Institute of Naval Architects, vol. 46, 1904.

ship. The primary function of the gyro is to oppose the *velocity* of rolling rather than—as is often stated—the moment causing the roll.

To prevent the ship from putting energy enough into the gyro to make $\varphi \geqq 90°$, energy is taken out by means of brake bands at the trunnions or by a dash-pot device acting against $\dot{\varphi}$: *i.e.*, the φ motion is damped. But this is not the sole condition for stability: the gyro must be pendulous—see § 101. Instead of attaching a weight as in i, we can mount the rotor casing with its centroid below the trunnion axis.

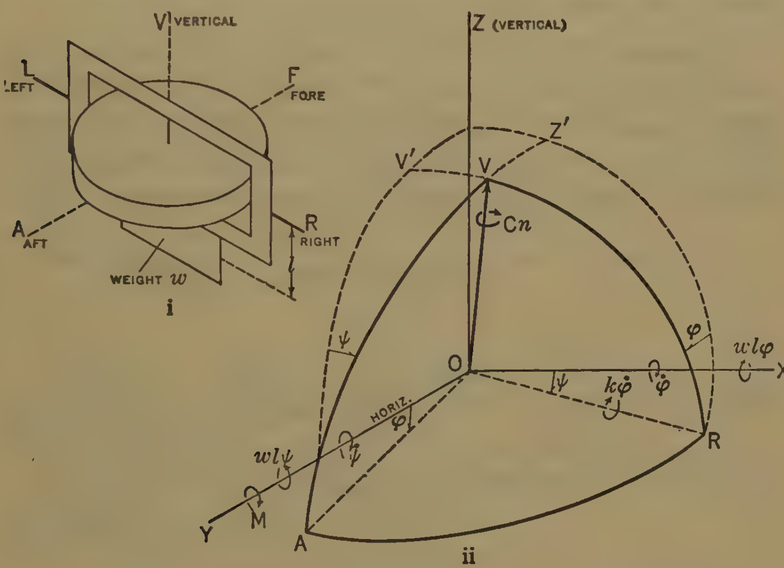


FIG. 69

98. Motion of the Gyro. Angles in Fig. 69 are assumed small, not over 10°. This will be correct enough when the stabilizer functions properly. The equations of large motions are quite intractable.

The moments of inertia of the rotor are

$$A \text{ about } OR, \quad B \text{ about } OA, \quad C \text{ about } OV.$$

The centroid of w is l below O in ii. The damping moment is $k\dot{\varphi}$ about OR. Since the momentum Cn is turned at the rates $\dot{\varphi}$ and $\dot{\psi}$, the equations of *small* motion are—the directions of $\dot{\varphi}$, $\dot{\psi}$ being positive—

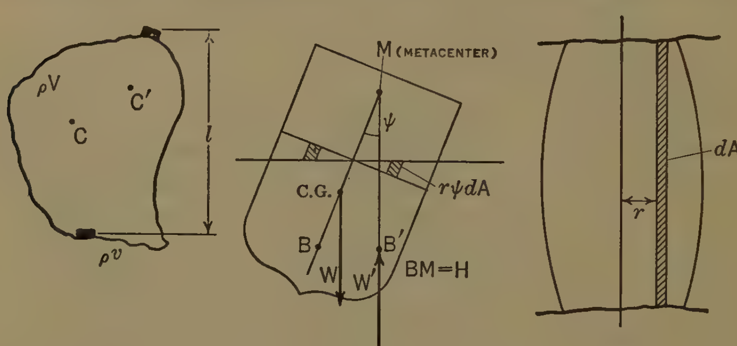
$$(1) \qquad -wl\varphi - k\dot{\varphi} = A\ddot{\varphi} - Cn\dot{\psi},$$

$$(2) \qquad -wl\psi + M = B\ddot{\psi} + Cn\dot{\varphi}.$$

In (1), (2), the yawing of the ship,[1] and second order terms, for example, $\dot{\varphi}\dot{\psi}$, are neglected.

Ex. 134. Prove that for large φ, ψ the moments of w are $wl \sin \varphi \cos \psi$ about OR, $-wl \sin \psi$ about OA.

99. Motion of the Ship. The integration of (1), (2) cannot be carried out until M is found. The ship is acted on by the righting moment: the moment of the couple formed by its weight W and the equal buoyant force passing through the metacenter. The metacenter lies at the intersection of the



I II III

Fig. 70

buoyant force for a small displacement of the ship from the vertical in a calm sea and the originally vertical axis of the ship, Fig. 70, II.

In I, C is the centroid of a mass of volume V, density ρ. When

[1] For an interesting study of yawing as the result of gyroscopic action see K. Suyehiro, *Yawing of ships*, Trans. Inst. Nav. Arch., 1920, pp. 93–101.

part of it, ρv, is shifted through a vertical distance l, the centroid of V moves to C' through a vertical distance L. By the principle of conservation of energy we have

$$\rho vgl = \rho VgL$$

or

$$vl = VL,$$

which is a purely geometric relation independent of weight.

In II the immersion volume is unaltered by rotation of the ship, because the weight of the displaced water equals the weight of the ship. The volume, however, is redistributed by the rotation, and its centroid moves from B (centroid of original immersion volume) to the new centroid B', on account of a wedge of immersion being shifted from left to right. III shows the horizontal water-line section of the ship.

If V is the displacement volume and $dA(r\psi)$ an element of a wedge, then as each element is shifted $2r$,

$$VH\psi = \int (dAr\psi)2r$$

or

$$VH = I,$$

where I is the moment of inertia of the water-line section about the longitudinal axis.

If h is the distance from the ship centroid to M, the righting moment is

$$-Wh\psi,$$

being negative because the sense of ψ is positive. There is also a pitching metacenter which we shall not consider.

There is also a moment on the ship due to distortion of the sea level by waves. Imagine waves coming from the *left*, broadside to the ship, which is assumed vertical for the present. The disturbing moment varies from zero when the ship is in a trough to zero when it is on a crest. As the change is continuous (smooth), we can assume for a first approximation that the

moment varies as a sine function, being

$$+P \sin pt;$$

positive because waves from the left turn the ship in the sense of $\dot{\psi}$. The angular fréquency of the waves is p, T in $pT = 2\pi$ being the period. The period of ordinary ocean waves is a few seconds and is less than the free period of the ship. The largest observed period[1] is 16.5 secs. for waves in the South Pacific Ocean; their height was 46 ft., length 760 ft., velocity 46 ft. per sec.

In his studies in wave action Froude[2] attempted to justify the assumption of a simple harmonic moment and showed that its consequences agree quite well with observation. The correct action is certainly more complex and ought to deal with the trochoidal wave form, the heaving of the ship, and with a succession of waves represented by a Fourier series. Froude took P as $Wh\theta$, where θ is the maximum slope of the wave shape —about 9°.

In addition to these moments and the reaction M of the gyro trunnions, there is a resistance to rolling that is proportional to $\dot{\psi}$, say

$$K\dot{\psi},$$

where K is to be found experimentally from the rate at which it damps the rolling; it averages from about 0.02 to 0.06 ft.-ton.

If I is the moment of inertia of the ship (not the water line section) about a longitudinal axis through the centroid of the ship, the equation of rolling is

$$(3) \qquad P \sin pt - M - Wh\psi - K\dot{\psi} = I\ddot{\psi}.$$

100. Ship without Gyro. The natural motion of the ship—no gyro, no waves, no damping—is given by the equation

$$-Wh\psi = I\ddot{\psi};$$

[1] Gaillard, *Wave Action*, Washington, D. C., 1904, p. 76.
[2] Trans. Inst. Nav. Arch., 1861, 1862.

the period is

$$T_0 = 2\pi\left(\frac{I}{Wh}\right)^{1/2},$$

where h ranges from 0.5 to 5 ft. and T_0 from 10 to 20 secs.

The natural period of the pendulous swing of the gyro when it is not spinning is

$$2\pi\left(\frac{A}{wl}\right)^{1/2}.$$

The free damped motion of the ship is given by

$$I\ddot{\psi} + K\dot{\psi} + Wh\psi = 0,$$

the solution of which is, as in § 77,

$$\psi = \psi_0 e^{-(Kt/2I)} \cos (qt - \alpha),$$

where

$$q = \left(\frac{Wh}{I} - \frac{K^2}{4I^2}\right)^{1/2};$$

the period of the free motion is

$$T_d = \frac{2\pi}{q}.$$

We have $T_d > T_0$ on account of damping. Observations of the rate at which the roll is extinguished give data from which K can be computed.

Ex. 135. Find K/I if one quarter of the roll is damped out in one period of 15 secs. See p. 138.

As in § 93, the forced oscillations are found from

$$I\ddot{\psi} + K\dot{\psi} + Wh\psi = P \sin pt$$

to be

$$\psi = a \sin (pt - \epsilon),$$

where

$$\tan \epsilon = \frac{Kp}{Wh - Ip^2},$$

$$a = \frac{P \sin \epsilon}{Kp}.$$

To keep a small the denominator of $\tan \epsilon$ should be considerably different from zero; any approach to resonance between the waves and the natural period of the ship is dangerous.

101. Stability of the Gyro. Eliminate M from (1) and (3); then put $B + I = J$ and, since wl is small, $Wh + wl = Wh$. Equations (1) and (2) become, with $Cn = N$,

$$(4) \qquad J\ddot{\psi} + K\dot{\psi} + Wh\psi + N\dot{\varphi} = P \sin pt,$$

$$(5) \qquad A\ddot{\varphi} + k\dot{\varphi} + wl\varphi - N\dot{\psi} = 0.$$

Differentiate (4) and substitute $\dot{\psi}$ from (5):

$$(6) \qquad \begin{aligned} JA\dddot{\varphi} + (Jk + KA)\dddot{\varphi} + (Jwl + Kk + WhA + N^2)\ddot{\varphi} \\ + (Kwl + Whk)\dot{\varphi} + Whwl\varphi = NPp \cos pt. \end{aligned}$$

When the right member of (6) is replaced by zero, the equation is satisfied by $\varphi = \epsilon^{\lambda t}$; substitution gives a quartic (fourth degree equation) in λ having the coefficients in (6). The most general form of λ is

$$\lambda = a + ib.$$

Hence

$$\varphi = e^{\lambda t} = e^{at}e^{ibt} = e^{at}(\cos bt + i \sin bt)$$

by De Moivre's theorem. If a is positive, φ increases with t and the motion is unstable. To avoid instability, the real part of λ must be negative.

Routh [1] has shown that if the λ quartic is written in the form

$$\lambda^4 + c_1\lambda^3 + c_2\lambda^2 + c_3\lambda + c_4 = 0,$$

the necessary and sufficient conditions for stability (real parts of roots negative) are

$$(7) \qquad c_1 > 0, \quad c_1c_2 > c_3, \quad (c_1c_2 - c_3)c_3 > c_1^2c_4, \quad c_4 > 0;$$

for a cubic, $c_1 > 0, \quad c_1c_2 > c_3, \quad c_3 > 0;$

for a quadratic, $c_1 > 0, \quad c_2 > 0.$

It is evidently necessary, although not sufficient, that all coefficients be positive. As c_4 cannot be positive unless Wh and wl have like signs, two gyroscopically coupled systems of

[1] Routh, *Advanced Rigid Dynamics*, § 300.

type (4) and (5) cannot be dynamically stable unless both are gravitationally stable or both gravitationally unstable. This shows why the centroid of the gyro of a ship stabilizer must lie below the trunnion axis.

102. Effect of the Gyro. In order to make all phases of the stabilizing or roll-quenching action clear, it will be necessary to examine some special cases. The effect of the gyro on the motion of the ship, *when there are no waves and no damping*, is found from (4) and (5) with $P = K = k = 0$. The elimination of φ gives an equation like (6), with $P = K = k = 0$ and φ replaced by ψ. The quartic resulting from the substitution of $\psi = e^{\lambda t}$ is

$$JA\lambda^4 + (Jwl + WhA + N^2)\lambda^2 + Whwl = 0. \tag{8}$$

Hence

$$2JA\lambda^2_{1,2} = -(Jwl + WhA + N^2) \pm \{(Jwl + WhA + N^2)^2 - 4JAWhwl\}^{1/2}.$$

The radical, being greater than its value when N is omitted, *i.e.*, greater than $\{(Jwl - WhA)^2\}^{1/2}$, can be written in the form

$$Jwl - WhA + \Delta,$$

where Δ is due to the presence of N. Hence

$$\begin{cases} \lambda_1^2 = -\dfrac{Wh}{J} - \dfrac{(N^2 - \Delta)}{2JA} = -p_1^2, \\[2ex] \lambda_2^2 = -\dfrac{wl}{A} - \dfrac{(N^2 + \Delta)}{2JA} = -p_2^2, \end{cases} \tag{9}$$

or

$$\lambda_1 = \pm ip_1, \qquad \lambda_2 = \pm ip_2.$$

Consequently the solution of the differential equation corresponding to (8) is

$$\psi = c_1e^{ip_1t} + c_2e^{-ip_1t} + c_3e^{ip_2t} + c_4e^{-ip_2t}.$$

Using De Moivre's theorem, we get after some reduction

$$\psi = a \sin (p_1 t + \epsilon_1) + b \sin (p_2 t + \epsilon_2), \tag{10}$$

whence the ship has two superposed harmonic motions.

From p. 173 the equation defining Δ is

$$\{(Jwl - WhA)^2 + N^4 + 2N^2(Jwl + WhA)\}^{1/2} = Jwl - WhA + \Delta.$$

Hence

$$\Delta > N^2$$

or say

$$\Delta = N^2 + \delta.$$

Hence, from (9),

$$\begin{cases} p_1^2 = \dfrac{Wh}{J} - \dfrac{\delta}{2JA}, \\ p_2^2 = \dfrac{wl}{A} + \dfrac{N^2}{JA} + \dfrac{\delta}{2JA}. \end{cases} \tag{11}$$

But from p. 171, the square of the natural angular frequency of the ship is

$$Wh/I,$$

and that of the gyro is

$$wl/A.$$

As I, J are practically the same, the effect of the gyro is to lengthen the natural period of roll (see(11)) of the ship, and to convert its other oscillation into a fast tremor much faster than the pendulous swing of the gyro.

Slowing the roll of the ship is advantageous in that it diminishes the harmonic accelerations. They are responsible for sea sickness, which results to a considerable extent from the alternating kinetic reactions on one's viscera. The yaw is also reduced.

It can easily be anticipated from energy considerations that the amplitude of the roll is not decreased, but we shall now prove it.

We can take as partial solutions of the equations (4) and (5), with $P = K = k = 0$,

$$\psi = re^{\lambda_1 t}, \qquad \varphi = Re^{\lambda_1 t};$$

there are of course other solutions corresponding to the remaining three λ's from (9). Substitution gives

$$Jr\lambda_1^2 + NR\lambda_1 + Whr = 0,$$
$$AR\lambda_1^2 - Nr\lambda_1 + wlR = 0,$$

or

$$(J\lambda_1^2 + Wh)r + N\lambda_1 R = 0,$$
$$(A\lambda_1^2 + wl)R - N\lambda_1 r = 0,$$

whence

$$\frac{R}{r} = \pm i\left(\frac{J\lambda_1^2 + Wh}{A\lambda_1^2 + wl}\right)^{1/2} = s,$$

with a similar ratio for the coefficients of the remaining three solutions.

By De Moivre's theorem the partial solutions of (4) and (5) are

$$(12)\qquad \begin{cases} \psi = r(\cos \lambda_1 t + i \sin \lambda_1 t), \\ \varphi = sr(-\sin \lambda_1 t + i \cos \lambda_1 t). \end{cases}$$

Since $s > 1$ it follows that the φ amplitudes are greater than the ψ; in fact s is so large as to make stops necessary to limit the swings of the gyro. When the gyro stops, its stabilizing action ceases.

Now (12) is a solution even when the last terms of (9) vanish, as they do when there is no gyro. Hence the undamped stabilizer does not decrease the amplitude of roll.

Equations (12) show that ψ and φ interchange form every time λt changes by 90°. Since this is true for every partial solution, it is true for the complete solutions; that is, ψ, φ always maintain a phase difference of 90°, or ψ and $\dot{\varphi}$ are always in the same phase.

103. The Damped Gyro. When the ship is undamped and there are no waves, $K = 0$, $P = 0$. With this substitution the ψ motion is defined by (6) when ψ is put for φ. Since $\psi = e^{\lambda t}$ satisfies the resulting equation, we have

$$(13)\qquad y = \alpha\lambda^4 + \beta\lambda^3 + \gamma\lambda^2 + \delta\lambda + \epsilon = 0,$$

where

$$\alpha = JA, \quad \beta = Jk, \quad \gamma = Jwl + WhA + N^2, \quad \delta = Whk, \quad \epsilon = Whwl.$$

If (13) has real roots, it is a simple matter to calculate them by Horner's method when the coefficients are numerical, *i.e.*, *after* the stabilizer has been designed. When the roots are complex they can be found by Ferrari's method, or by the following process. In $f(\lambda) = 0$ put $\lambda = p + iq$, where p, q are real. Then

$$f_1(p, q) + if_2(p, q) = 0.$$

Hence $f_1 = 0$ and $f_2 = 0$, from which p (or q) can be eliminated and the real roots of the result computed by Horner's method.

Ex. 136. Show that the roots of $x^4 = -1$ are $\pm 0.71(1 \pm i)$; that $p = 0$, $q = \pm 1$ for $x^4 = 1$; that the roots of $x^4 + 4x^3 + 6x^2 + 4x + 5 = 0$ are $\pm i$, $-2 \pm i$.

The action of a stabilizer on a given ship rolled by given waves depends on the selection of the best values of N, k, wl/A. It is often no easy matter to decide what is meant by 'best,' the answer being generally a compromise among conflicting requirements. The influence of N, k, wl/A on the period and amplitude of the ship has been worked out in detail by Noether.[1] We shall show the effect of k by a simple geometric method merely to illustrate a possible line of attack in such cases.

Separate (13) into

$$y_1 = \alpha\lambda^4 + \beta\lambda^3 + \gamma\lambda^2, \tag{14}$$

$$-y_2 = \delta\lambda + \epsilon, \tag{15}$$

which intersect when $y_1 - y_2 = 0$, *i.e.* at the *real* roots of (13).

To plot (14), note that two roots are zero. The other two, which satisfy the equation

$$\alpha\lambda^2 + \beta\lambda + \gamma = 0,$$

are complex or real when

$$\beta^2 \lesseqgtr 4\alpha\gamma$$

or when

$$Jwl + WhA + N^2 \gtreqless \frac{Jk^2}{4A}. \tag{16}$$

[1] Klein and Sommerfeld, *Theorie des Kreisels*, IV, pp. 810–833.

From the conditions for stability, p. 172, we get

(17) $$Jwl + WhA + N^2 > WhA$$

and

$$N^2 > 1.$$

For very heavy damping (large k), (16), (17) give

(18) $$Jk^2 > 4WhA.$$

The two derivatives of (14) are

(19) $$y_1' = 4\alpha\lambda^3 + 3\beta\lambda^2 + 2\gamma\lambda,$$

(20) $$y_1'' = 12\alpha\lambda^2 + 6\beta\lambda + 2\gamma.$$

The graphs of (14), (15) are drawn roughly in Fig. 71 for light, moderate, and heavy damping of the gyro.

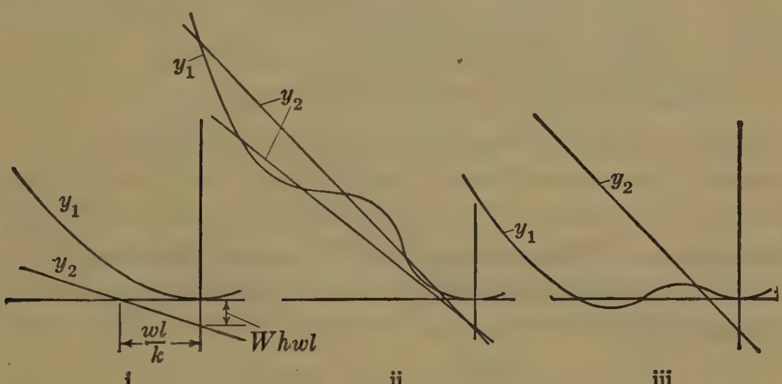


FIG. 71

i. The line y_2 gets progressively steeper as k increases. For small k the curve y_1 has no horizontal tangent except at the origin, and no points of inflection. There are no intersections and no real roots.

ii. When k is sufficiently large, (20) can have two points of inflection without making (19) vanish. There is a geometric possibility of two or four real roots.[1]

[1] For the proof that four are physically possible, see Klein and Sommerfeld, *Theorie des Kreisels*, IV, p. 817.

iii. For a very large k, $\beta^2 > 4\alpha\gamma$; hence (14) can have two real roots, (19) can show three horizontal tangents, and (20) two points of inflection. But as y_2 is steep there will be only two intersections. This result is surprising, as one would expect a continuous passage, with increasing k, from zero to two to four real roots.

104. Light Damping. All roots are complex, Fig. 71, i, and of the forms

$$-\lambda_1 \pm iq_1, \quad -\lambda_2 \pm iq_2,$$

where the real parts are negative because (17), p. 177, are satisfied.

The solution as in § 102 is

$$\begin{aligned}\psi &= c_1e^{-\lambda_1 t}(\cos q_1t + i\sin q_1t) + c_2e^{-\lambda_1 t}(\cos q_1t - i\sin q_1t)\\ &\qquad + c_3e^{-\lambda_2 t}(\cos q_2t + i\sin q_2t) + c_3e^{-\lambda_2 t}(\cos q_2t - i\sin q_2t)\\ &= c_5e^{-\lambda_1 t}\sin(q_1t + \epsilon_1) + c_6e^{-\lambda_2 t}\sin(q_2t + \epsilon_2),\end{aligned}$$

showing that the oscillations are exponentially damped. As k is small the periods are not much different from those in the case of $k = 0$, § 102. The equation for the real part of the root (p on p. 173) should be studied for ways of increasing λ_1, λ_2.

105. Moderate Damping. When two roots are real and two complex they are

$$-r_1, \quad -r_2, \quad -\lambda \pm iq$$

and the solution is easily found to be

$$\psi = c_1e^{-r_1t} + c_2e^{-r_2t} + c_3e^{-\lambda t}\sin(qt + \epsilon).$$

The first two terms vanish in course of time. When there are four real roots two of them are likely to be small, Fig. 71, ii, and the amplitudes are only slowly reduced.

106. Heavy Damping. One real root in iii is small and denotes a slowly extinguished aperiodic swing. On the other hand, the second real root is very large, which means that for β/α (sum of the roots of (13), p. 175) of a given order of magnitude the

real parts of the complex roots are correspondingly small; in this case the periodic motion is slowly damped. Heavy damping of the gyro therefore is not effective in damping the ship. The heavily damped gyro diminishes its own oscillations until its stabilizing action nearly stops. When $k = \infty$, (13) has the two roots

$$\lambda = \pm i\sqrt{\frac{Wh}{J}},$$

and the ship has merely its natural period; infinite k means of course that the gyro is clamped and out of action.

Heavy damping reacts on the ship and may cause pitching.

107. Forced Oscillations. Wave action was not taken into account in the preceding discussion. To study the effect of the waves we shall find the forced oscillations from (4), (5), p. 172. We infer from the case examined in § 93 that the forced oscillations of the ship will be synchronous with the waves; therefore try

$$\psi = \rho R \sin (pt + r),$$
$$\varphi = R \cos (pt + s),$$

as solutions of (4) and (5). The sine is suggested in ψ because the wave force varies as the sine; the cosine is tried in φ because it was found on p. 175 that ψ and φ differ by 90°.[1] Substitution gives

$$(1)\quad \rho Ra \sin (pt+r) + \rho Rb \cos (pt+r) - Rc \sin (pt+s) = P \sin pt,$$

$$(2)\quad \alpha \cos (pt+s) - \beta \sin (pt+s) - \rho c \cos (pt+r) = 0,$$

where

$$(3)\quad a = Wh - Jp^2, \qquad b = Kp, \qquad c = Np,$$

$$(4)\quad \alpha = wl - Ap^2, \qquad \beta = kp.$$

As (1) and (2) are to be true for any t, they are satisfied by

[1] The use of the operator D will suggest itself to readers familiar with differential equations; however, the method of "main strength and awkwardness," such as that in the text, is often the only one available. Rotating vectors are convenient in many cases; see V. Bush, *Operational Circuit Analysis*, 1929.

$t = 0$ and $pt = 90°$; hence

(5) $$pa \sin r + pb \cos r - c \sin s = 0,$$

(6) $$\alpha \cos s - \beta \sin s - \rho c \cos r = 0,$$

(7) $$pa \cos r - pb \sin r - c \cos s = \frac{P}{R},$$

(8) $$-\alpha \sin s - \beta \cos s + \rho c \sin r = 0.$$

To find ρ, solve (6) for $\rho c \cos r$, (8) for $\rho c \sin r$, square and add:

(9) $$\rho^2 N^2 p^2 = (wl - Ap^2)^2 + k^2 p^2.$$

To save space, the details of the following reductions are omitted.

Solve (5) for $c \sin s$, (7) for $c \cos s$, substitute these values in (6) and (8) and call the results

(10), (11),

from which

$$\tan r = \frac{\beta^2 + b\rho^2}{\alpha - a\rho^2},$$

$$\tan s = \frac{a\beta + \alpha b}{a\alpha - b\beta - c^2}.$$

Multiply (10) by $\cos r$, (11) by $\sin r$, and add; then multiply (10) by $\sin r$, (11) by $\cos r$, and subtract. The result of a somewhat lengthy elimination is

(12) $$\frac{c^2(\alpha^2 + \beta^2)P^2}{R^2} = [\alpha c^2 - a(\alpha^2 + \beta^2)]^2 + [\beta c^2 + b(\alpha^2 + \beta^2)]^2.$$

Equations (9) and (12) show the difficulty of selecting N, k, wl/A to make R and ρR as small as possible without conflicting with the desirable lengthening of the period of roll. Further discussion is unnecessary, as the Schlick stabilizer has been superseded by the Sperry or *active type* stabilizer.

108. The Active Type Stabilizer. From (12), p. 175, ψ and $\dot{\varphi}$ have the same form and therefore reach their minimum values simultaneously. Damping will of course produce a phase differ-

ence between them, and the difference between r and s in the forced oscillations, p. 180, is not exactly 90°. But unless k is large the gyro will precess at its least velocity *near* the least value of ψ. The ship must therefore roll an appreciable amount from the vertical before $\dot{\varphi}$ and the gyroscopic moment produced by it become large enough to be effective.

Sperry conceived the idea of precessing the gyro by means of an automatically controlled motor so that the largest value of the gyroscopic moment occurs when the ship is rolling fastest, that is, when the ship is vertical. A stabilizer of this type is called *active;* it is superior to the Schlick passive type in that the gyro is considerably lighter and can be designed to counteract the roll at its inception, *before appreciable amplitudes are built up.*

In dealing with the Schlick gyro, we did not take into account the *initial stages* of the motion caused by wave action. Let the ship be vertical and at rest when the waves start; the forced oscillations will not exist alone and the amplitudes will not be constant until enough time has elapsed for the extinction of the free oscillations. During this time the amplitudes vary in

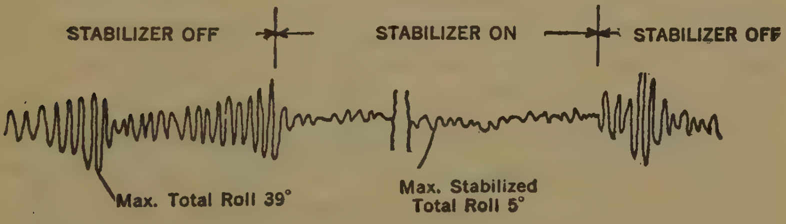


FIG. 72

different ways, depending on the ratio of the free and forced periods and on the damping. Steady motion is seldom maintained in practice because of the erratic or, rather, complex character of the waves. Graphs drawn by a Sperry roll recorder, operating a pencil on a moving strip of paper, show that the roll is cumulative as in Fig. 72. Consequently if each increment

of amplitude is counteracted by the stabilizer the effect of the waves will be reduced to a negligible minimum.

For clearness a *plan* view of the control gyro is shown in Fig. 73; the stabilizing gyro is drawn in elevation. Neither

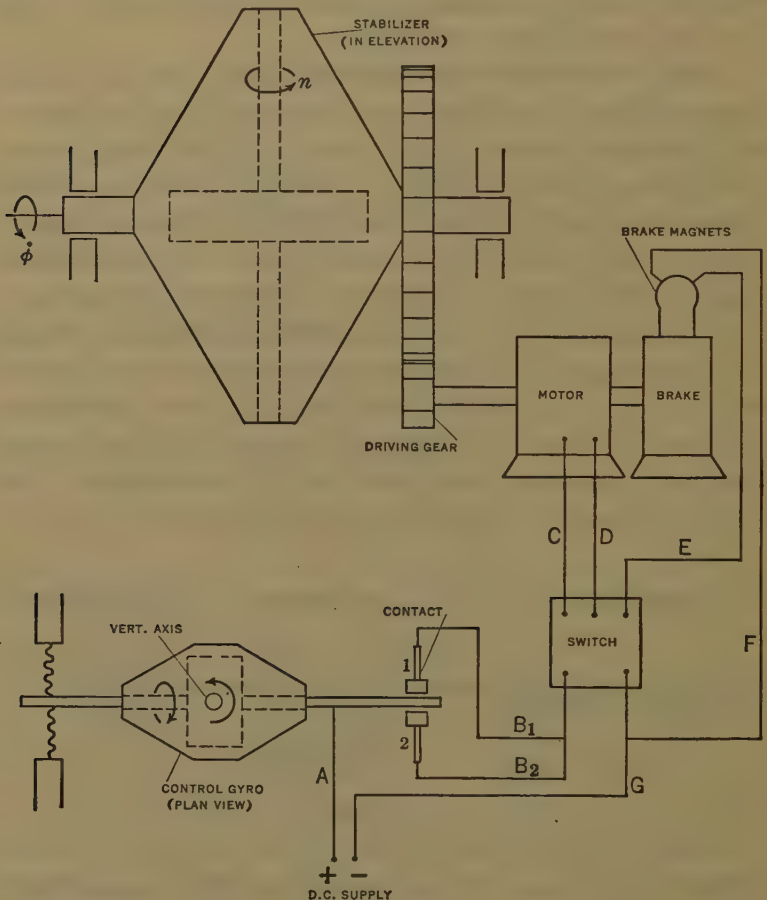


FIG. 73

gyro is pendulous. The diagram is schematic in order to emphasize the principles of operation.

The main gyro, axle normally vertical, is on trunnions athwart the ship. The small control gyro has its axle horizontal and

athwart the ship, its case being mounted on a *vertical* axis. Both rotors are driven by A.C. motors not shown.

The switch reverses the current through the motor, when necessary, and also opens or closes the circuit through the brake magnets that control the brake. When the current is zero (off), the brake holds the motor shaft but gradually loosens its grip as the current increases.

Let the ship begin to roll clockwise as viewed from aft. When the roll is large enough the control gyro precesses counterclockwise, viewed from above, and making contact at 1 closes the switch circuit AB_1G. The switch automatically closes circuit AB_1CDEFG and passes full current in series through the motor and the brake magnets. As soon as the magnets are energized they release the brake and allow the motor to run up to speed and to drive the main gyro in the direction $\dot{\varphi}$ shown. This precession induces a counterclockwise moment (viewed from aft) on the trunnion bearings and acts against the rolling *velocity* of the ship.

The clockwise roll of the ship also precesses the gyro in the sense of $\dot{\varphi}$. When the roll is fast enough to overspeed and thus to drive the motor, the motor acts as a generator and, by driving a counter E.M.F. against the line, reduces the effective current supply. As the motor and the brake magnets are in series, the magnets permit the brake to grip and so to retard the motor. This keeps the motor speed from fluctuating more than 5%.

When the roll is sufficiently retarded the precession of the control gyro becomes too small to overcome the stiffness of the centralizing springs, which then break the contact at 1. During the reverse roll contact occurs at 2, the switch circuit is AB_2G and the main circuit AB_2DCEFG. This reverses the motor.

If the main gyro is not vertical when the velocity of roll is reduced below the sensitivity of the control gyro, it will remain inclined until the end of that roll. The next roll will not be properly damped but the following one will be.

109. Theory of the Sperry Stabilizer. Let the gyroscopic moment on the ship be M; M opposes the velocity of roll and changes sign with it. As the gyro is not pendulous and B is small compared with Cn in (2), p. 168, $M = Cn\dot{\varphi}$. It is assumed in (2) that φ is small. Actually, however, φ reaches 60°, in which case

(1) $$M = Cn\dot{\varphi}\cos\varphi.$$

For simplicity we shall take M as constant with $\varphi = 60°$; this errs on the safe side. For large φ there is also a moment $(Cn\sin\varphi)\dot{\varphi}$ about a vertical axis; the forced yaw resulting from this alternating moment is very small on account of the enormous moment of inertia of the ship and the great resistance of the water.

For one swing from extreme left to extreme right the equation of motion of the ship is, like (3), p. 170,

(2) $$I\ddot{\psi} + K\dot{\psi} + Wh\psi = M + P\sin pt,$$

in which M is left-handed, *i.e.* against $Wh\psi$ and $\dot{\psi}$. Equation (2) can be resolved into the elements

(3) $$I\ddot{\psi}_1 + K\dot{\psi}_1 + Wh\psi_1 = M,$$

(4) $$I\ddot{\psi}_2 + K\dot{\psi}_2 + Wh\psi_2 = P\sin pt,$$

where

$$\psi_1 + \psi_2 = \psi,$$

ψ_1 referring to oscillations damped by K and M, and ψ_2 to the forced oscillations built up by the waves. When the damping produced by M is adjusted to balance the roll increment arising from P, the resultant (2) of the motions (3) and (4) is an oscillation of constant amplitude.

It is easily verified (see § 77) that the complete solution of (3) is

$$\psi_1 = \frac{M}{Wh} + Ae^{-(Kt/2I)}\sin\left(\frac{2\pi t}{T} + \epsilon\right),$$

T being the free period of the ship.

If $\psi_1 = a$ at $t = 0$, and $\psi_1 = -b$ at $t = T/2$, the *sum* of the results of substitution is

$$\frac{2M}{Wh} = a - b - A(1 - e^{-(KT/4I)}) \sin \epsilon \tag{5}$$
$$= \Delta \text{ radians.}$$

Hence Δ, the actual decrement, $a - b$, minus the decrement from water resistance, is the roll-quenching capacity of the stabilizer, that is, Δ is produced by the stabilizer in a calm, frictionless sea. Fig. 74 shows schematically how the stabilizer functions.

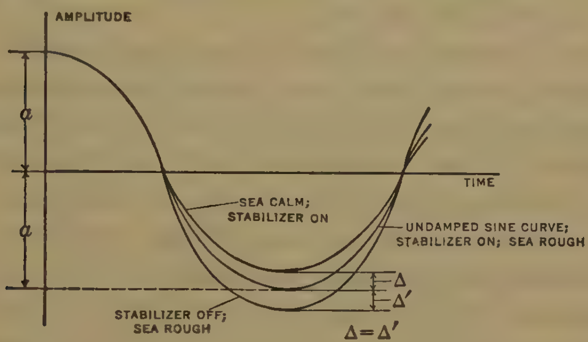


FIG. 74

But Δ', the increment of ψ_2, is found from the rolling-up record, Fig. 72. Making $\Delta = \Delta'$ prevents the increase of amplitude. As Δ' is not the same for all swings, the largest one is selected. Consequently the stabilizer produces a net decrease of amplitude every time an increment is less than the largest. The decrease continues until the amplitudes are less than two or three degrees.

The stabilizer equation, from (1) and (5), that is,

$$\Delta = \frac{2Cn\dot{\varphi} \cos 60}{Wh} \tag{6}$$

is inexact on account of the assumptions of constant φ and $\dot{\varphi}$ and small ψ, but it is satisfactory when modified by an empirical constant. It takes 0.2 sec. for the control gyro to operate and

0.6 sec. for the motor to get up to speed. Therefore $\dot{\varphi}$ must be somewhat greater (about 40%) than its average value

$$2 \times 60^\circ \times (\pi/180)/T \text{ rad. per sec.}$$

Before the end of the roll is reached the rolling velocity becomes too small to actuate the control gyro, which then returns to its central position and puts on the brake to stop the precession.

Illustrative values are:

$$W = 4500 \text{ tons}, \qquad h = 1.62 \text{ ft.}, \qquad T = 12 \text{ secs.}$$
$$C = 620{,}000 \text{ lb.-ft.}^2, \qquad n = 1050 \text{ r.p.m.}$$
$$\dot{\varphi} = 0.575 \text{ rad. per sec.}, \qquad \Delta = 6^\circ.$$

110. The Vertical-Gyro Monorail Car. A gyro stabilizer similar to the Schlick can be used to maintain a monorail vehicle upright, provided it is mechanically feasible to satisfy the conditions for stability. To keep the last term of (6), p. 172, positive, the gyro needs to be top-heavy or gravitationally unstable to correspond to this state of the car. But making both Wh and wl negative makes the coefficient of $\dot{\varphi}$ negative. As it is out of the question to have a negative K, the only alternative is to reverse k, *i.e.*, to *accelerate* the precession. Physically, this means that damping both car and gyro takes too much energy out of the system to allow both centroids to rise at the expense of their potential energy. Equation (6), p. 172, takes the form

$$JA\ddddot{\varphi} + (KA - Jk)\dddot{\varphi} + (N^2 - Jwl - WhA - Kk)\ddot{\varphi} + (Whk - Kwl)\dot{\varphi} + Whwl\varphi = 0.$$

Stability requires all coefficients to be positive—see § 101. Hence

$$Whk > Kwl, \qquad KA > Jk, \qquad \text{or} \qquad Wh/J > wl/A,$$

whence the car must have a faster natural period than the pendulous period of the gyro. Since $J > A$, the condition $KA > Jk$ gives $k < K$. Devices for accelerating the precession have been patented by Sherl and by Schilovsky.[1]

[1] See Scherl's Brit. patent 21843, 1908; Schilovsky's U. S. patent 1041680, 1912.

A vertical-gyro monorail car cannot make left and right turns with equal ease unless it is equipped with special devices not yet perfected. Consider the case of a car making a left turn and inclined toward the left, which is of course necessary for the comfort of passengers; let the rotor spin left-handed when viewed from above. The leftward horizontal component of angular momentum of the gyro is rotated by the turning, and produces a righting moment on the car—see Huntington's Rule, p. 50—which might raise the car to the vertical and perhaps beyond. During a right turn the gyroscopic moment is also right-handed, which is rather better than in the first case.

111. The Brennan Monorail Car. In the Brennan stabilizer the gyros—two for symmetry—are mounted on horizontal axles as in Fig. 75, which shows an early simplified form. The frame holding them is pivoted on a longitudinal axis. The rotor spins are opposite, and the rotor cases are geared together to precess symmetrically in opposite directions about the vertial.

Wheels a, d are keyed to the rotor shafts but b, c are loose. A, B, C, D are horizontal shelves on which a, b, c, d run when necessary.

When the car is tipped toward the right, the following action takes place. The gyros give the frame an enormous dynamical moment of inertia about P and tend to hold it upright. Shelf A is thus brought up against wheel a. This tips the frame, but so slightly as to be negligible except in a quantitative analysis. The wheel a therefore rolls into the paper; the gearing carries d, now free from its shelf, also into the paper. The rolling of a is assisted by the precession resulting from the upward push of A. The precession then reacts to increase the push on A; this righting moment on the car is helped by the left-handed moment on the frame arising from the precession of the right-hand gyro.

The friction accompanying the rolling of a is necessary for stability. As in the case of the vertical stabilizer on p. 110, it accelerates the precession.

The righting moment continues to act until the car is tipped to the left of the vertical. Being top-heavy, it continues tipping until C touches c. The upward push on c precesses the axle

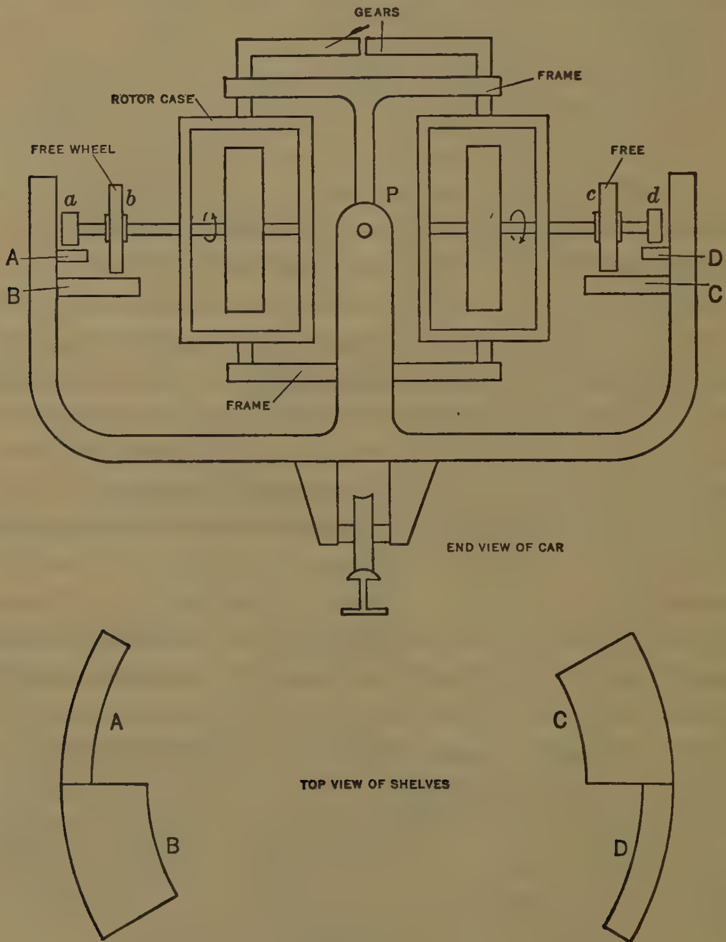


FIG. 75

toward the reader until the rotor axles overshoot their collinear position and bring D against d. The action then repeats, and the gyros and car execute damped oscillations.[1]

[1] For the mathematical theory see Cousins, Engineering, Nov. 21, 1913.

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